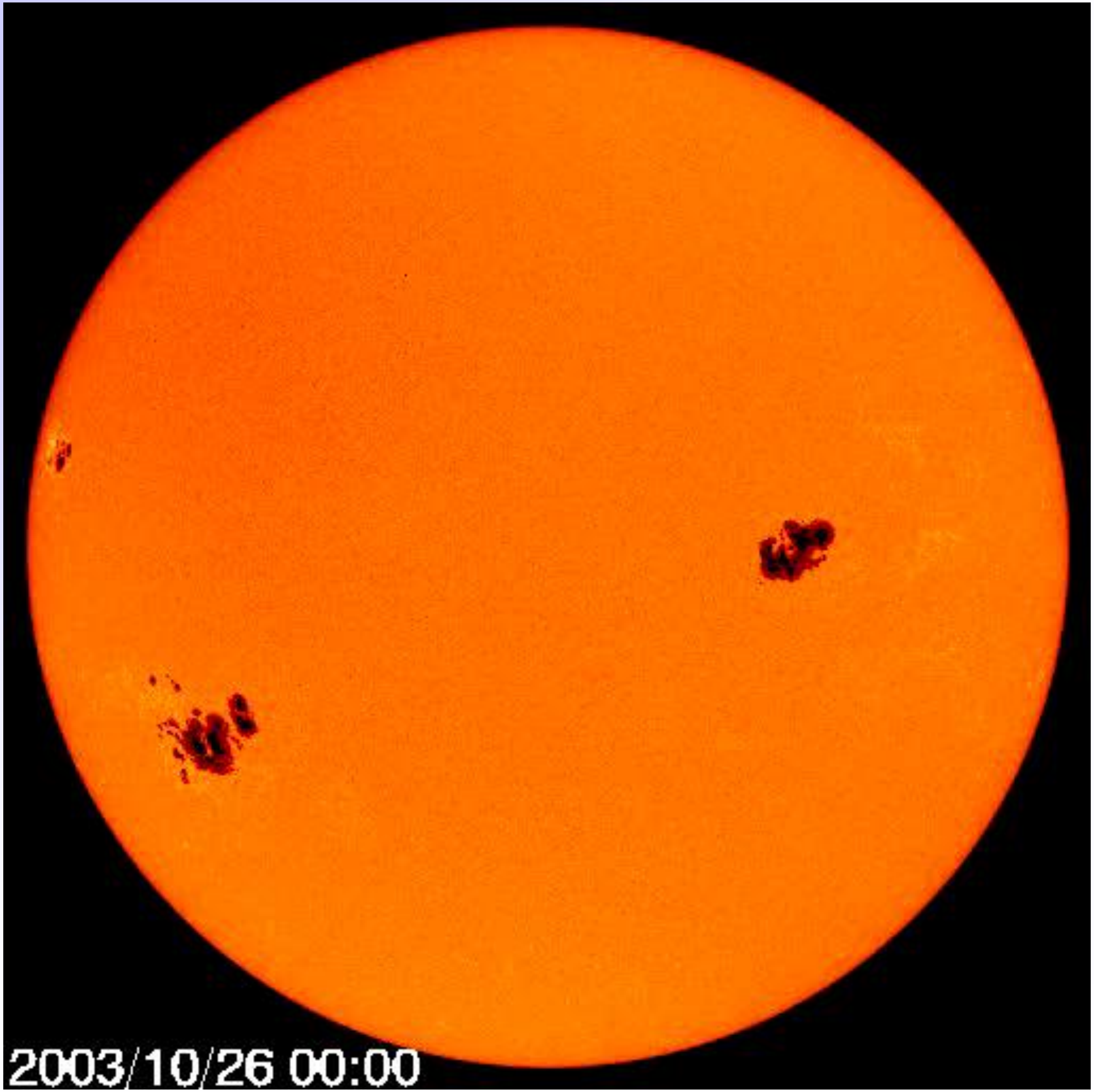
13ème cours de Mécanique Analytique (9/12/2010)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- 2.11 La méthode d'Hamilton-Jacobi

$$K = H + \frac{\partial F}{\partial t} \qquad (2.111)$$

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0 \qquad (2.112)$$

$$S(q, P, t) = F_2(q, P, t)$$

$$p_i = \frac{\partial S}{\partial q_i} \qquad (2.113)$$

$$\boxed{H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0} \qquad (2.114)$$

équation d'Hamilton-Jacobi

- 2.11 La méthode d'Hamilton-Jacobi

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$$ (2.115)

$$S = S(q_1, ..., q_f, \alpha_1, ..., \alpha_{f+1}, t)$$ (2.116) *solution complète*

si S est une solution de cette équation, alors $S + \alpha_i$ aussi!

$$S = S_0(q_1, ..., q_f, \alpha_1, ..., \alpha_f, t) + \alpha_{f+1}$$ (2.117)

$$S = S(q_1, ..., q_f, \alpha_1, ..., \alpha_f, t)$$ (2.118) *solution complète*

$$P_i = \alpha_i \qquad (i = 1, ..., f)$$ (2.119)

• 2.11 La méthode d'Hamilton-Jacobi

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i} \quad (2.120)$$

$$Q_i = \frac{\partial S}{\partial P_i} = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i} = \beta_i \quad (2.121)$$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \quad (2.122)$$

$$q_i = q_i(\alpha,\ \beta,\ t) \quad (2.123)$$

$$dtm\left(\frac{\partial^2 S}{\partial q_i \partial \alpha_j}\right) \neq 0 \quad (2.124)$$

fonction principale d'Hamilton

$$S(q, \alpha, t)$$

$$p_i = p_i(\alpha,\ \beta,\ t) \quad (2.125)$$

- 2.11 La méthode d'Hamilton-Jacobi

$S(q, \alpha, t)$?

$p_i \dot{q}_i - H = \cancel{P_i \dot{Q}_i} - \cancel{K} + dS/dt$

$dS/dt = p_i \dot{q}_i - H = L$

$S = \int L dt$

• 2.11 La méthode d'Hamilton-Jacobi

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0 \qquad (2.126)$$

$$S(q_i, \alpha_i, t) = W(q_i, \alpha_i) - \alpha_1 t \qquad (2.127)$$

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1 \qquad (2.128)$$

équation d'Hamilton-Jacobi réduite

… équation d'Hamilton-Jacobi.

$$\left\{ \begin{array}{l} p_i = \dfrac{\partial W(q, \alpha')}{\partial q_i} \\ Q'_i = \dfrac{\partial W}{\partial P'_i} = \dfrac{\partial W}{\partial \alpha'_i} \end{array} \right. \qquad (2.129)$$

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i} \qquad (2.120)\ (2.121)$$

$$Q_i = \frac{\partial S}{\partial P_i} = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i} = \beta_i$$

• 2.11 La méthode d'Hamilton-Jacobi

$$H(q_i, p_i) = \alpha_1' \qquad (2.130)$$

$$\begin{cases} p_i = \dfrac{\partial W}{\partial q_i} & (2.129) \\ Q_i' = \dfrac{\partial W}{\partial P_i'} = \dfrac{\partial W}{\partial \alpha_i'} \end{cases}$$

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1' \qquad (2.131)$$

$$K' = \alpha_1$$

$$K' = K'(P') = \alpha_1$$

$$\dot{P}_i' = -\frac{\partial K'}{\partial Q_i'} = 0 \rightarrow P_i' = \alpha_i' \qquad (2.132)$$

et

$$\dot{Q}_i' = \frac{\partial K'}{\partial \alpha_i'} = 1 \qquad i = 1 \qquad (2.133)$$

$$= 0 \qquad i \neq 1$$

- 2.11 La méthode d'Hamilton-Jacobi

$$\dot{P}_i' = -\frac{\partial K'}{\partial Q_i'} = 0 \rightarrow P_i' \quad = \quad \alpha_i' \qquad (2.132)$$

et

$$\dot{Q}_i' = \frac{\partial K'}{\partial \alpha_i'} = 1 \qquad i \quad = \quad 1 \qquad (2.133)$$

$$= 0 \qquad i \quad \neq \quad 1$$

$$Q_1' = t + \beta_1 = \frac{\partial W}{\partial \alpha_1}$$

$$Q_i' = \beta_i' = \frac{\partial W}{\partial \alpha_i'} \qquad (i \neq 1) \qquad (2.134)$$

• 2.12 Méthode de séparation des variables + exemples

$$\boxed{H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}\cancel{, t}\right) + \frac{\partial S}{\partial t} = 0} \quad (2.114)$$

équation d'Hamilton-Jacobi

$$S(q, \alpha, t) = S_0(\alpha, t) + W(q, \alpha) \quad (2.137)$$

$$H\left(q, \frac{\partial W}{\partial q}\right) + \frac{\partial S_0}{\partial t} = 0 \quad (2.138)$$

$$\frac{\partial S_0}{\partial t} = -\alpha_1 \quad (2.139) \qquad H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1 \quad (2.140)$$

$$S_0 = -\alpha_1 t \quad (2.141)$$

équation d'Hamilton-Jacobi réduite

… équation d'Hamilton-Jacobi.

- 2.12 Méthode de séparation des variables + exemples

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

$$S(q,\alpha,t) = S_0(\alpha,t) + W(q,\alpha)$$ (2.137)

$$S_0 = -\alpha_1 t$$ (2.141)

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha_1$$

- 2.12 Méthode de séparation des variables + exemples

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha_1$$

$$W = \sqrt{2m\alpha_1}\int\sqrt{1-\frac{m\omega^2 q^2}{2\alpha_1}}dq$$

$$Q_i = \frac{\partial S}{\partial P_i} = \frac{\partial S(q,\alpha,t)}{\partial \alpha_i} = \beta_i \quad (2.121)$$

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial W}{\partial \alpha_1} - t = \sqrt{\frac{2m}{\alpha_1}}\int\frac{dq}{\sqrt{1-\frac{m\omega^2 q^2}{2\alpha_1}}} - t$$

• 2.12 Méthode de séparation des variables + exemples

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial W}{\partial \alpha_1} - t = \sqrt{\frac{m}{2\alpha_1}} \int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha_1}}} - t$$

$$Q' = \frac{\partial W}{\partial \alpha_1} = t + \beta_1 = \frac{1}{\omega} \arcsin q\sqrt{\frac{m\omega^2}{2\alpha_1}}$$

$$q = \sqrt{\frac{2\alpha_1}{m\omega^2}} \sin \omega(t + \beta_1)$$

$$\begin{aligned} p = \frac{\partial W}{\partial q} &= \sqrt{2m\alpha_1 - m^2\omega^2 q^2} \\ &= \sqrt{2m\alpha_1} \cos \omega(t + \beta_1) \end{aligned}$$

- 2.12 Méthode de séparation des variables + exemples

$$p = \frac{\partial W}{\partial q} = \sqrt{2m\alpha_1 - m^2\omega^2 q^2}$$
$$= \sqrt{2m\alpha_1}\cos\omega(t+\beta_1)$$

$$\alpha_1 = \frac{p_0^2 + m\omega^2 q_0^2}{2m}$$

$$\mathrm{tg}\,(\omega\beta_1) = m\omega\frac{q_0}{p_0}$$

- 2.12 Méthode de séparation des variables + exemples

$$H\left(q_2, \ldots, q_f; \gamma, \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_f}\right) = \alpha_1 \qquad (2.142)$$

$$W = W_1(q_1, \alpha) + W'(q_2, \ldots, q_f; \alpha) \qquad (2.143)$$

$$p_1 = \gamma = \frac{\partial W_1}{\partial q_1} \qquad (2.144)$$

$$W_1 = \gamma q_1 \qquad (2.145)$$

$$W = W' + \gamma q_1 \qquad (2.146)$$

- 2.12 Méthode de séparation des variables + exemples

$$W = \sum_{i=1}^{f} W_i(q_i, \alpha) = W_1(q_1, \alpha) + \sum_{i=2}^{f} \alpha_i q_i \qquad (2.147)$$

$$H\left(q_1, \frac{\partial W_1}{\partial q_1}, \alpha_2, \ldots, \alpha_n\right) = \alpha_1 \qquad (2.148)$$

$$f(q_j, p_j)$$

$$W = W_j(q_j, \alpha) + W'(q_i, \alpha) \qquad (2.149)$$

$$H\left(q_i, \frac{\partial W'}{\partial q_i}, f\left(q_j, \frac{\partial W_j}{\partial q_j}\right)\right) = \alpha_1 \qquad (2.150)$$

• 2.12 Méthode de séparation des variables + exemples

$$H\left(q_i, \frac{\partial W'}{\partial q_i}, f\left(q_j, \frac{\partial W_j}{\partial q_j}\right)\right) = \alpha_1 \qquad (2.150)$$

$$f\left(q_j, \frac{\partial W_j}{\partial q_j}\right) = g\left(q_i, \frac{\partial W'}{\partial q_i}, \alpha_1\right) \qquad (2.151)$$

$$\begin{cases} f\left(q_j, \dfrac{\partial W_j}{\partial q_j}\right) = \alpha_j \\ g\left(q_i, \dfrac{\partial W'}{\partial q_i}, \alpha_1\right) = \alpha_j \end{cases} \qquad (2.152)$$

• 2.12 Méthode de séparation des variables + exemples

Exemple 1 :

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + V(r)$$

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2\right] + V(r) = \alpha_1$$

$$W = W_\phi(\phi) + W_2(r) = \alpha_\phi \phi + W_2(r)$$

$$\text{puisque } p_\phi = \frac{\partial W_\phi}{\partial \phi} = \alpha_\phi \qquad (\alpha_\phi = \text{constante})$$

• 2.12 Méthode de séparation des variables + exemples

$$\frac{1}{2m}\left[\left(\frac{dW_2}{dr}\right)^2 + \frac{\alpha_\phi^2}{r^2}\right] + V(r) = \alpha_1$$

$$\frac{dW_2}{dr} = \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\phi^2}{r^2}}$$

$$W = \int \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\phi^2}{r^2}}\, dr + \alpha_\phi \phi$$

• 2.12 Méthode de séparation des variables + exemples

$$W = \int \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\phi^2}{r^2}}\, dr + \alpha_\phi \phi$$

$$Q'_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1}$$

$$Q'_i = \beta'_i = \frac{\partial W}{\partial \alpha'_i} \qquad (i \neq 1)$$

(2.134)

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \int \frac{m\, dr}{\sqrt{2m(\alpha_1 - V) - \frac{\alpha_\phi^2}{r^2}}}$$

$$\beta_\phi = \frac{\partial W}{\partial \alpha_\phi} = -\int \frac{\alpha_\phi\, dr}{r^2 \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\phi^2}{r^2}}} + \phi$$

• 2.12 Méthode de séparation des variables + exemples

Exemple 2 :

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) - V(r)$$

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}) + V(r)$$

$$\frac{1}{2m}\left(\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right) + V(r) = E$$

$$W(r,\theta,\phi) = W_\phi(\phi) + W'(r,\theta)$$

$$W_\phi = \alpha_\phi \phi$$

• 2.12 Méthode de séparation des variables + exemples

$$\frac{1}{2m}\left\{\left(\frac{\partial W'}{\partial r}\right)^2+\frac{1}{r^2}\left[\left(\frac{\partial W'}{\partial \theta}\right)^2+\frac{\alpha_\phi^2}{\sin^2\theta}\right]\right\}+V(r)=E$$

$$W'(r,\theta)=W_r(r)+W_\theta(\theta)$$

$$\left(\frac{dW_\theta}{d\theta}\right)^2+\frac{\alpha_\phi^2}{\sin^2\theta}=\alpha_\theta^2$$

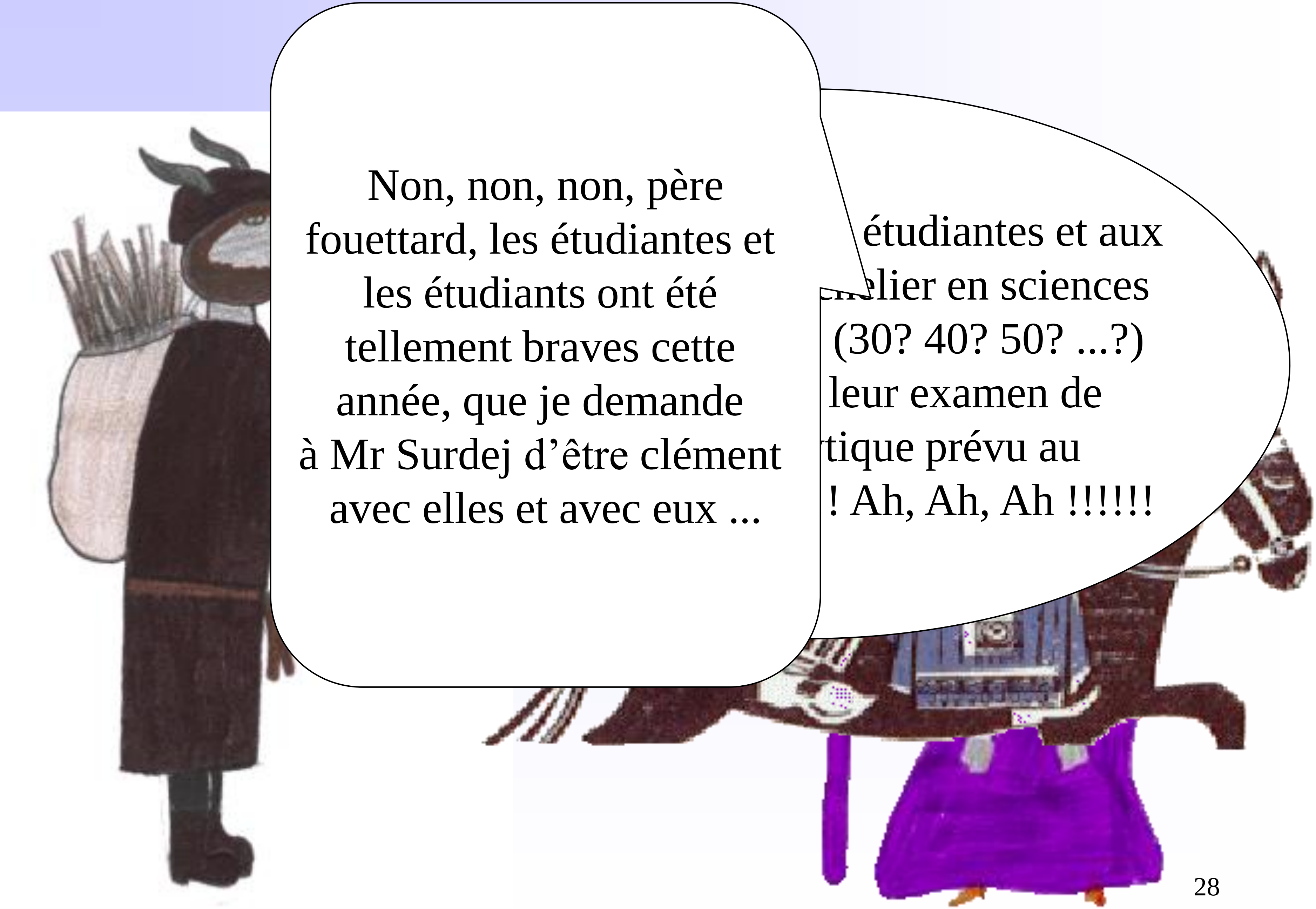
$$\frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2+\frac{\alpha_\theta^2}{r^2}\right]+V(r)=E$$

$$W_\theta=\int\sqrt{\alpha_\theta^2-\frac{\alpha_\phi^2}{\sin^2\theta}}\,d\theta$$

$$W_r=\int\sqrt{2m[E-V(r)]-\frac{\alpha_\theta^2}{r^2}}\,dr$$

$$\alpha_\phi=p_\phi$$

$$\alpha_\theta^2=p_\theta^2+\left(\frac{p_\phi}{\sin\theta}\right)^2=m^2r^4(\dot\theta^2+\sin^2\theta\,\dot\phi^2)$$


Non, non, non, père fouettard, les étudiantes et les étudiants ont été tellement braves cette année, que je demande à Mr Surdej d'être clément avec elles et avec eux ...
étudiantes et aux
elier en sciences
(30? 40? 50? ...?)
leur examen de
tique prévu au
! Ah, Ah, Ah !!!!!!

Principe variationnel d'Hamilton ...
et formation de mirages atmosphériques ...