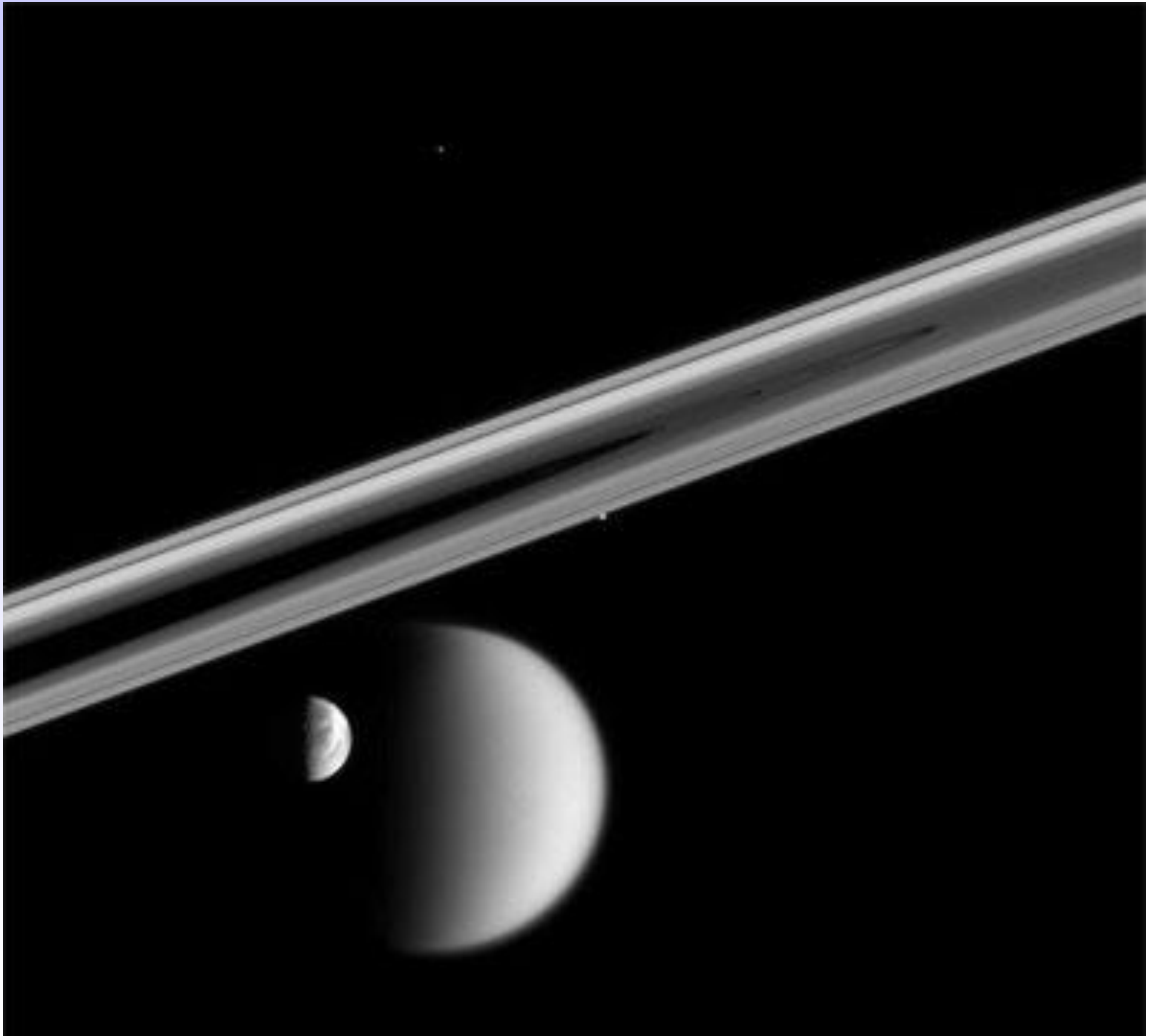
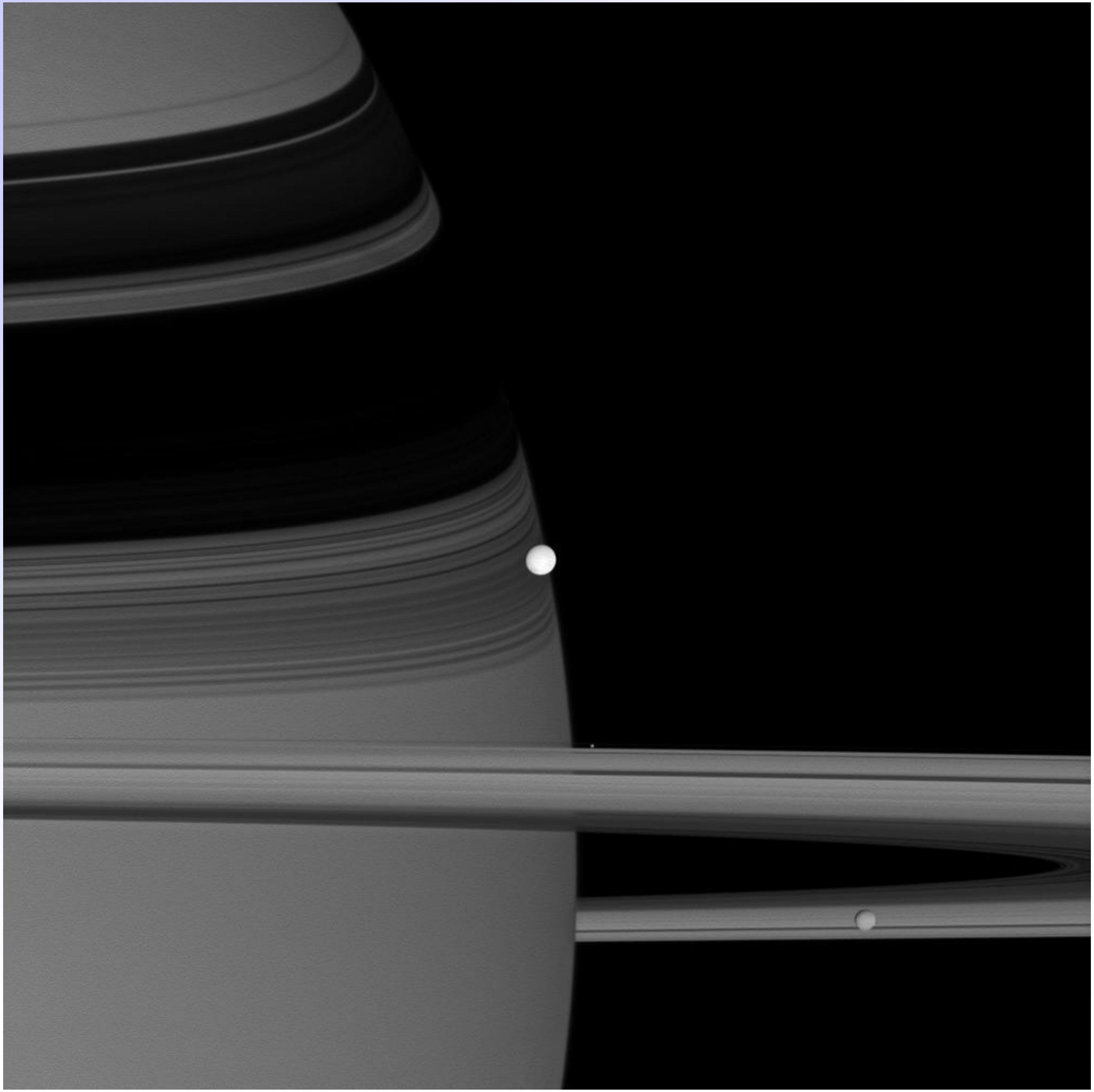
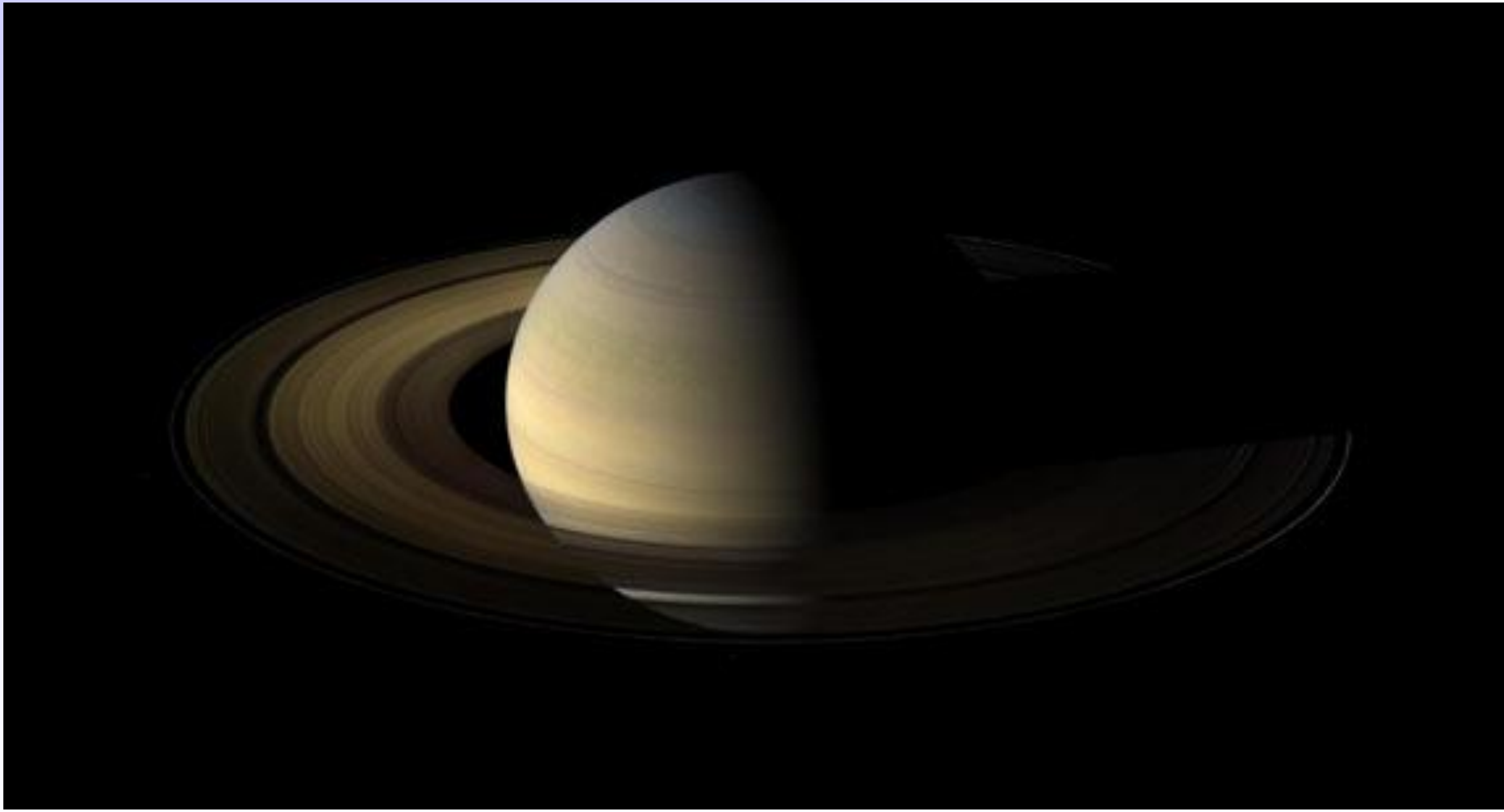
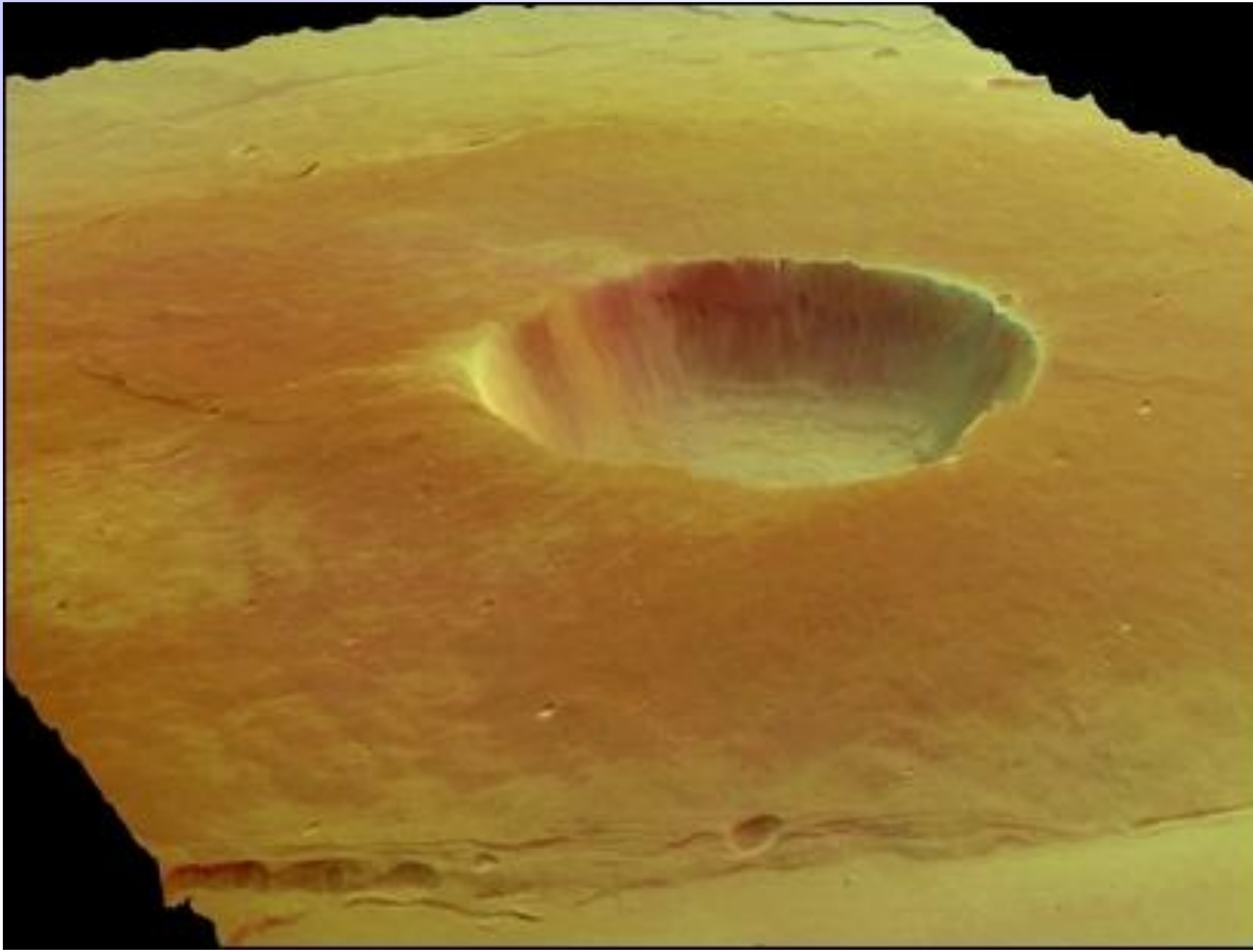
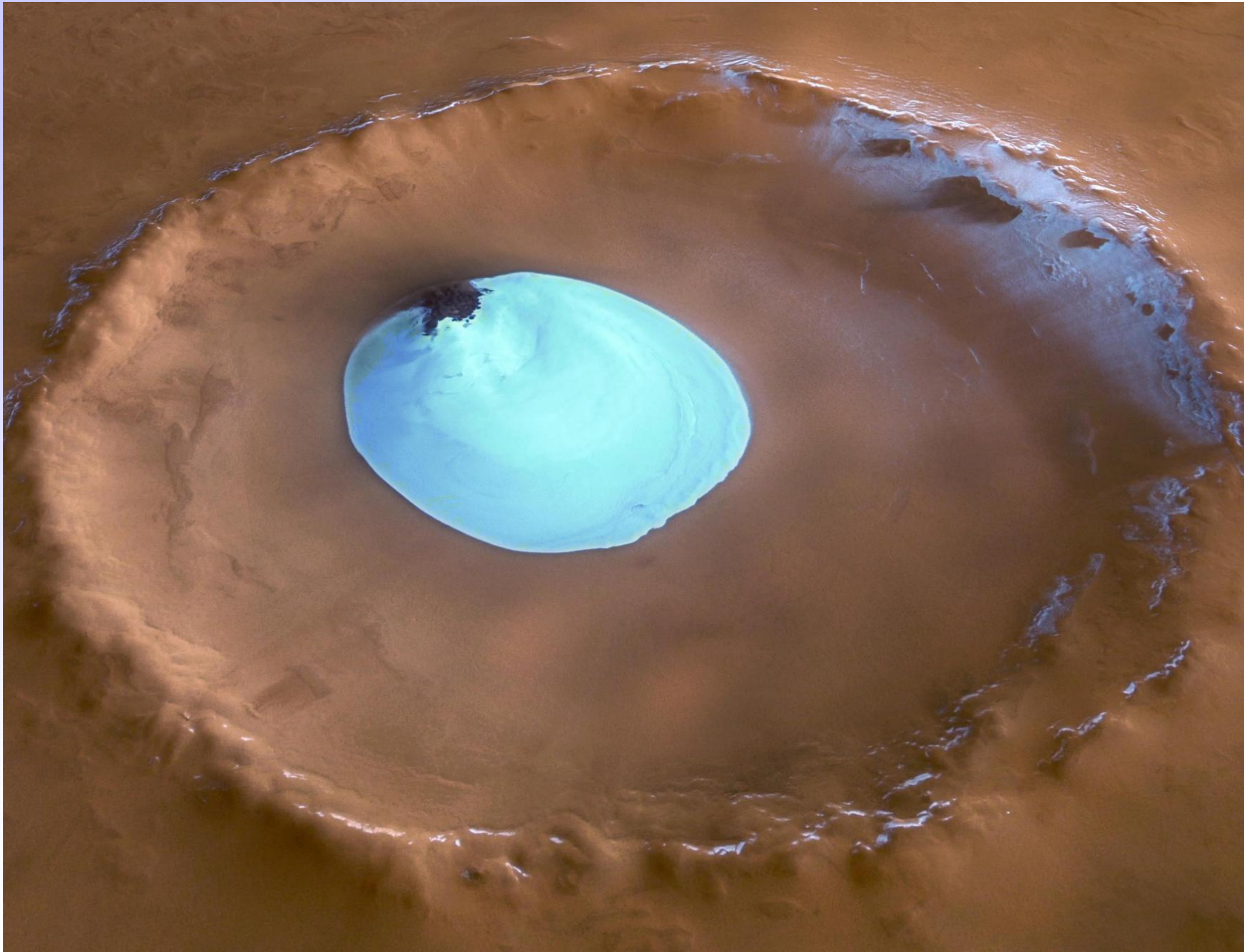
12ème cours de Mécanique Analytique (2/12/2010)

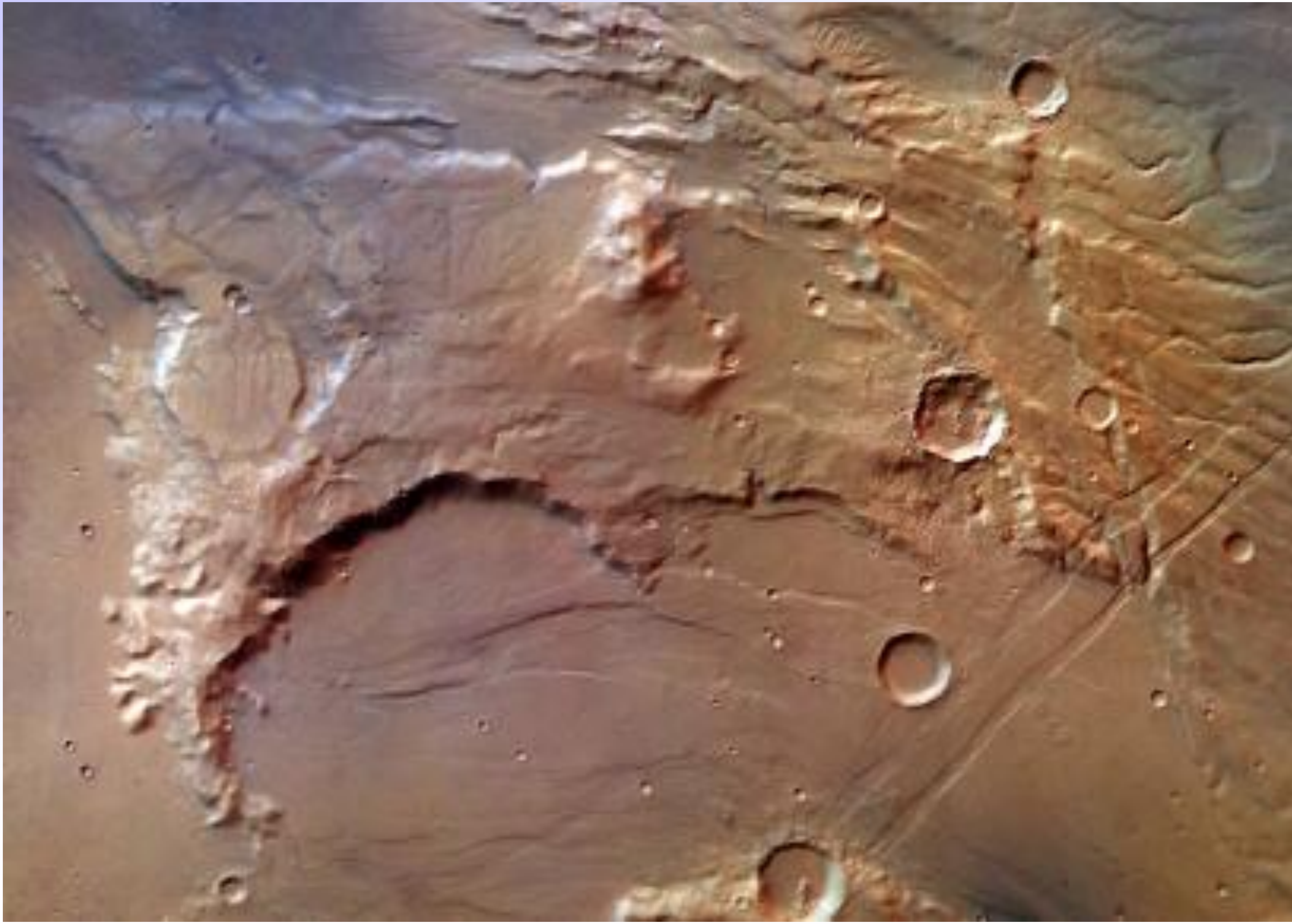


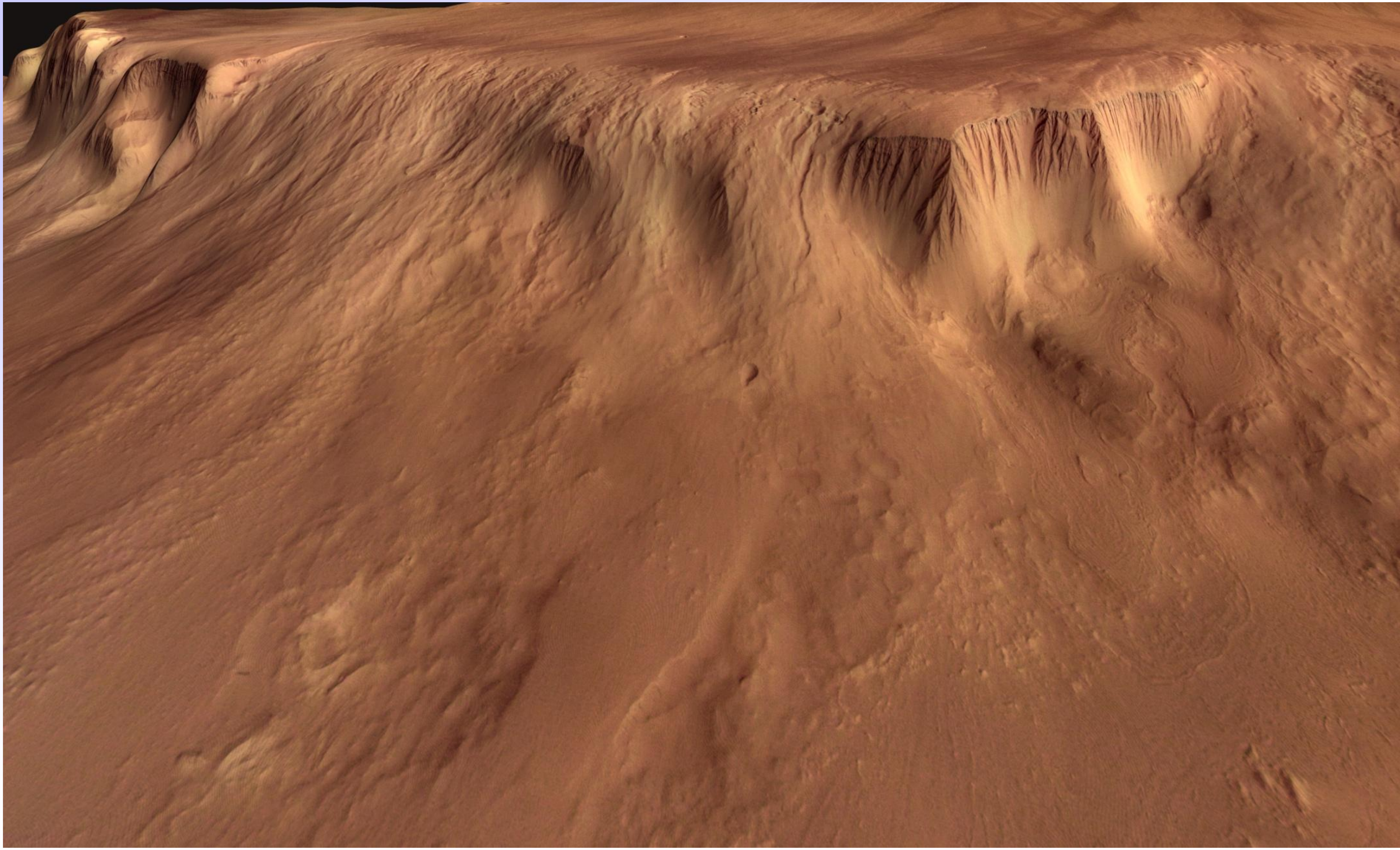












• 2.4 Transformations canoniques

$$\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = P_i(q, p, t) \end{cases} \quad (2.19)$$

$$\boxed{\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \qquad (i = 1, \ldots, f)} \quad (2.20)$$

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))\, dt = 0$$

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t))\, dt = 0 \quad (2.21)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \quad (2.22)$$

$$F = F_1(q, Q, t)$$

$$F = F_2(q, P, t) - Q_i P_i$$

$$F = q_i p_i + F_3(p, Q, t)$$

$$F = q_i p_i - Q_i P_i + F_4(p, P, t)$$

• 2.5 Exemples de transformations canoniques

a) $F_2(q, P) = q_i P_i$ (2.33)

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \qquad (2.34a)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i \qquad (2.34b)$$

$$K = H \qquad (2.34c)$$

b) $F_1 = q_k Q_k$ (2.35)

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i \qquad (2.36a)$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i \qquad (2.36b)$$

$$K(Q, P) = H(-P, Q) \qquad (2.36c)$$

• 2.6 L'approche symplectique des transformations canoniques

vecteur colonne η à $2f$ éléments $\eta_i = q_i \quad , \quad n_{i+f} = p_i \quad i \leq f$ (2.45)

le vecteur colonne $\frac{\partial H}{\partial \eta}$ $\left(\frac{\partial H}{\partial \eta}\right)_i = \frac{\partial H}{\partial q_i} \quad , \quad \left(\frac{\partial H}{\partial \eta}\right)_{i+f} = \frac{\partial H}{\partial p_i} \quad i \leq f$ (2.46)

J la matrice carrée $2f \times 2f$ $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ (2.47)

$$\dot{\eta} = J \frac{\partial H}{\partial \eta} \quad (2.48)$$

• 2.6 L'approche symplectique des transf. canoniques

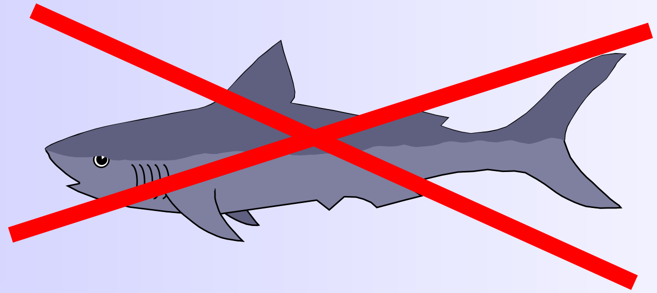
$$\dot{\zeta} = M J \tilde{M} \frac{\partial H}{\partial \zeta} \quad (2.55)$$

$$\dot{\zeta} = J \frac{\partial H}{\partial \zeta} \quad (2.56)$$

$$M J \tilde{M} = J \quad (2.57a)$$

$$\tilde{M} J M = J \quad (2.57b)$$

- 2.7 Les crochets de Poisson

[]

$$\boxed{[u,v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}} \quad (2.58)$$

$$[u,v]_\eta = \frac{\widetilde{\partial u}}{\partial \eta} J \frac{\partial v}{\partial \eta} \quad (2.59)$$

Si $u_i = \frac{\tilde{\partial} u}{\partial \eta_i}$ *et* $v_k = \frac{\partial v}{\partial \eta_k}$

$= u_i \, J_{ik} \, v_k$

$= u_i \, [\delta_{i,k-f} \, \Pi_{[1\leq i\leq f, f+1\leq k\leq 2f]} - \delta_{i-f,k} \, \Pi_{[f+1\leq i\leq 2f, 1\leq k\leq f]}] v_k$

• 2.7 Les crochets de Poisson

matrice crochet de Poisson

désignée par $[\eta, \eta]$ dont l'élément (ℓm) est $[\eta_\ell, \eta_m]$

$$[\eta, \eta]_\eta = J \qquad (2.61)$$

$$\zeta = \zeta(\eta, t)$$

$$[\zeta, \zeta]_\eta = \frac{\partial \zeta}{\partial \eta} J \widetilde{\frac{\partial \zeta}{\partial \eta}} \qquad (2.62)$$

$$[\zeta, \zeta]_\eta = M J \tilde{M} \qquad (2.63)$$

$$[\zeta, \zeta]_\eta = J \qquad (2.64)$$

$$[\zeta, \zeta]_\zeta = J \qquad (2.65)$$

• 2.7 Les crochets de Poisson

$$[u,v]_\eta = \widetilde{\frac{\partial u}{\partial \eta}} J \frac{\partial v}{\partial \eta} \qquad (2.59)$$

$$\frac{\partial v}{\partial \eta} = \tilde{M} \frac{\partial v}{\partial \zeta} \qquad (2.66)$$

$$\widetilde{\frac{\partial u}{\partial \eta}} = \widetilde{\tilde{M}\frac{\partial u}{\partial \zeta}} = \widetilde{\frac{\partial u}{\partial \zeta}} M \qquad (2.67)$$

$$[u,v]_\eta = \widetilde{\frac{\partial u}{\partial \eta}} J \frac{\partial v}{\partial \eta} = \widetilde{\frac{\partial u}{\partial \zeta}} M J \tilde{M} \frac{\partial v}{\partial \zeta} \qquad (2.68)$$

$$\boxed{[u,v]_\eta = \widetilde{\frac{\partial u}{\partial \zeta}} J \frac{\partial v}{\partial \zeta} = [u,v]_\zeta} \qquad (2.69)$$

• 2.7 Les crochets de Poisson

$$
\begin{aligned}
[u,u] &= 0 && (2.70a)\\
[u,v] &= -[v,u] && (2.70b)\\
[au+bv,w] &= a[u,w] + b[v,w] && (2.70c)\\
[uv,w] &= [u,w]v + u[v,w] && (2.70d)\\
[u,[v,w]] &+ [v,[w,u]] + [w,[u,v]] = 0 && (2.70e)
\end{aligned}
$$

$$u_i \equiv \frac{\partial u}{\partial \eta_i} \quad , \quad v_{ij} = \frac{\partial^2 v}{\partial \eta_i \partial \eta_j}$$

$$[u,v] = u_i J_{ij} v_j$$

$$[u,[v,w]] = u_i J_{ij} [v,w]_j = u_i J_{ij} (v_k J_{k\ell} w_\ell)_j$$

• 2.7 Les crochets de Poisson

$$[u,[v,w]] = u_i J_{ij} [v,w]_j = u_i J_{ij} (v_k J_{k\ell} w_\ell)_j$$

$$[u,[v,w]] = u_i J_{ij} (v_k J_{k\ell} w_{\ell j} + v_{kj} J_{k\ell} w_\ell) \quad (2.71)$$

$$J_{ij} J_{k\ell} u_i v_k w_{\ell j}$$

$$[v,[w,u]] = v_k J_{k\ell} (w_j J_{ji} u_i)_\ell$$

$$J_{ji} J_{k\ell} u_i v_k w_{j\ell}$$

$$(J_{ij} + J_{ji}) J_{k\ell} u_i v_k w_{\ell j} = 0 \quad (2.72)$$

• 2.7 Les crochets de Poisson

$$[u, u] = 0 \quad (2.70a)$$

$$[u, v] = -[v, u] \quad (2.70b)$$

$$[au + bv, w] = a[u, w] + b[v, w] \quad (2.70c)$$

$$[uv, w] = [u, w]v + u[v, w] \quad (2.70d)$$

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \quad (2.70e)$$

$$[u, v] \rightarrow \frac{1}{i\hbar}(uv - vu)$$

$$\hbar = \frac{h}{2\pi}$$

- **2.8** Formulation de la mécanique hamiltonienne dans le langage des crochets de Poisson

$$g(q,p,t)$$

$$\frac{dg}{dt}=\frac{\partial g}{\partial q_i}\dot{q}_i+\frac{\partial g}{\partial p_i}\dot{p}_i+\frac{\partial g}{\partial t}=\frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i}+\frac{\partial g}{\partial t}$$

$$\boxed{\frac{dg}{dt}=[g,H]+\frac{\partial g}{\partial t}} \quad (2.73)$$

$$\frac{dg}{dt}=\frac{\partial g}{\partial \eta}\dot{\eta}+\frac{\partial g}{\partial t}=\frac{\partial g}{\partial \eta}J\frac{\partial H}{\partial \eta}+\frac{\partial g}{\partial t} \quad (2.74)$$

$$\begin{cases}\dot{q}_i=[q_i,H]\\ \dot{p}_i=[p_i,H]\end{cases} \quad (2.75)$$

$$\dot{\eta}=[\eta,H]=J\frac{\partial H}{\partial \eta} \quad (2.76)$$

$$\frac{dH}{dt}=\frac{\partial H}{\partial t} \quad (2.77)$$

- 2.8 [Formulation de la mécanique hamiltonienne]

$$\frac{\partial g}{\partial t} = [H, g] \quad (2.78)$$

$$[H, g] = 0 \quad (2.79)$$

$$w = [u, v]$$

$$\begin{aligned} \frac{dw}{dt} &= [w, H] + \frac{\partial w}{\partial t} \\ &= \frac{\partial}{\partial t}[u, v] + [[u, v], H] \end{aligned}$$

- 2.8 [Formulation de la mécanique hamiltonienne]

$$\begin{aligned}\frac{dw}{dt} &= [w, H] + \frac{\partial w}{\partial t} \\ &= \frac{\partial}{\partial t}[u, v] + [[u, v], H]\end{aligned}$$

$$\begin{aligned}\frac{dw}{dt} &= \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] + [u, [v, H] + [v, [H, u]] \\ &= \left[\frac{\partial u}{\partial t} + [u, H], v\right] + \left[u, \frac{\partial v}{\partial t} + [v, H]\right] \\ &= \left[\frac{du}{dt}, v\right] + \left[u, \frac{dv}{dt}\right] \qquad (2.80)\end{aligned}$$

• 2.8 [Formulation de la mécanique hamiltonienne]

$$\begin{aligned}\frac{dw}{dt} &= \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] + [u,[v,H]] + [v,[H,u]] \\ &= \left[\frac{\partial u}{\partial t} + [u,H], v\right] + \left[u, \frac{\partial v}{\partial t} + [v,H]\right] \\ &= \left[\frac{du}{dt}, v\right] + \left[u, \frac{dv}{dt}\right] \qquad (2.80)\end{aligned}$$

$$\frac{du}{dt} = 0, \text{ et } \frac{dv}{dt} = 0$$

$$\frac{d}{dt}[u, v] = 0 \qquad (2.81)$$

Théorème de ?

- 2.9 Transformations canoniques infinitésimales et intégrales premières

- 2.9 Trans. can. infinit. et intégrales premières

$$F_2(q, P) = q_i P_i \qquad (2.33)$$

$$S_E(q, P) = q_i P_i \qquad (2.82)$$

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \qquad (2.34a)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i \qquad (2.34b)$$

$$K = H \qquad (2.34c)$$

$$\begin{cases} q_i = Q_i \\ p_i = P_i \\ K = H \end{cases} \qquad (2.83)$$

• 2.9 Trans. can. infinit. et intégrales premières

ε $\sigma(q,P)$

$$S(q,P,\varepsilon) = S_E + \varepsilon\sigma(q,P) + O(\varepsilon^2) \quad (2.84)$$

$$\sigma(q,P) = \left.\frac{\partial S}{\partial \varepsilon}\right|_{\varepsilon=0} \quad (2.85)$$

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \quad (2.34a)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i \quad (2.34b)$$

$$K = H \quad (2.34c)$$

$$\begin{cases} Q_i = \dfrac{\partial S}{\partial P_i} = q_i + \varepsilon\dfrac{\partial \sigma}{\partial P_i} + O(\varepsilon^2) \\ p_j = \dfrac{\partial S}{\partial q_j} = P_j + \varepsilon\dfrac{\partial \sigma}{\partial q_j} + O(\varepsilon^2) \end{cases} \quad (2.86)$$

• 2.9 Trans. can. infinit. et intégrales premières

$$\begin{cases} Q_i = \dfrac{\partial S}{\partial P_i} = q_i + \varepsilon \boxed{\dfrac{\partial \sigma}{\partial P_i}} + O(\varepsilon^2) \\ p_j = \dfrac{\partial S}{\partial q_j} = P_j + \varepsilon \boxed{\dfrac{\partial \sigma}{\partial q_j}} + O(\varepsilon^2) \end{cases} \quad (2.86)$$

$$\begin{cases} \delta q_i = Q_i - q_i = \varepsilon \dfrac{\partial \sigma(q,p)}{\partial p_i} \\ \delta p_j = P_j - p_j = -\varepsilon \dfrac{\partial \sigma(q,p)}{\partial q_j} \end{cases} \quad (2.87)$$

$$\boxed{[u,v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}} \quad (2.58)$$

$$\begin{cases} \delta q_i = \varepsilon\,[q_i, \sigma(q,p)] \\ \delta p_j = \varepsilon\,[p_j, \sigma(q,p)] \end{cases} \quad (2.88)$$

• 2.9 Trans. can. infinit. et intégrales premières

$$\begin{cases} \delta q_i = \varepsilon\,[q_i, \sigma(q,p)] \\ \delta p_j = \varepsilon\,[p_j, \sigma(q,p)] \end{cases} \qquad (2.88)$$

$$\delta\eta = \varepsilon J \frac{\partial \sigma(\eta)}{\partial \eta} = \varepsilon\,[\eta, \sigma(\eta)] \qquad (2.89)$$

$$S(q, P, \varepsilon) = S_E + \varepsilon\sigma(q, P) + O(\varepsilon^2) \qquad (2.84)$$

$$S(q, P, dt) = S_E + H(q,p)\,dt \qquad (2.90)$$

$\delta q_i \equiv dq_i$ et $\delta p_j \equiv dp_j$

$$\begin{cases} dq_i = [q_i, H]\,dt \\ dp_j = [p_j, H]\,dt \end{cases} \qquad (2.91)$$

- 2.9 Trans. can. infinit. et intégrales premières

$$\delta_\sigma f(q,p) = \sum_{k=1}^{f}\left(\frac{\partial f}{\partial q_k}\delta q_k + \frac{\partial f}{\partial p_k}\delta p_k\right) \qquad (2.92)$$

$$= \sum_{k=1}^{f}\left(\frac{\partial f}{\partial q_k}\frac{\partial \sigma}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial \sigma}{\partial p_k}\right)\varepsilon = \varepsilon\,[f,\sigma]$$

$$\frac{df}{dt} = [f,H] \qquad (2.93)$$

$$\left\{\begin{array}{l}\delta q_i = Q_i - q_i = \varepsilon\dfrac{\partial\sigma(q,p)}{\partial p_i} \\ \delta p_j = P_j - p_j = -\varepsilon\dfrac{\partial\sigma(q,p)}{\partial q_j}\end{array}\right. \qquad (2.87)$$

$$\delta_f H = \varepsilon\,[H,f] \qquad (2.94)$$

$$\delta_H f = [f,H]\,dt \qquad (2.95)$$

• 2.9 Trans. can. infinit. et intégrales premières

exemples simples :

a) $\vec{r}\,' = \vec{r} + \vec{a}$ (2.96)

$$S(\vec{r}, \vec{p}\,') = (\vec{r} + \vec{a}) \cdot \vec{p}\,' = \vec{r} \cdot \vec{p}\,' + \vec{a} \cdot \vec{p}\,' \quad (2.97)$$

$$\begin{cases} Q_k = \dfrac{\partial S}{\partial P_k} \quad \rightarrow \quad \vec{r}\,' = \vec{r} + \vec{a} & (k = 1, 2, 3) \\ p_k = \dfrac{\partial S}{\partial q_k} \quad \rightarrow \quad \vec{p}\,' = \vec{p} & \end{cases} \quad (2.98)$$

$$H = \frac{|\vec{p}|^2}{2m} \quad (2.99)$$

• 2.9 Trans. can. infinit. et intégrales premières

$$\sigma = \left.\frac{\partial S}{\partial a}\right|_{a=0} = \hat{a} \cdot \vec{p}$$

(2.100)

$$\delta_f H = \varepsilon \, [H, f]$$ (2.94)

⇓

$$[H, \sigma] = 0$$ (2.101)

⇓

$$\delta_H f = [f, H] \, dt$$ (2.95)

⇐

$$\frac{d\sigma}{dt} = 0$$ (2.102)

• 2.9 Trans. can. infinit. et intégrales premières

b)

$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

$$S(x,y,z,p'_x,p'_y,p'_z) = (x\cos\theta + y\sin\theta)p'_x + (-x\sin\theta + y\cos\theta)p'_y + zp'_z$$

(2.103)

$$\begin{cases} x' = \dfrac{\partial S}{\partial p'_x} = x\cos\theta + y\sin\theta \\ y' = \dfrac{\partial S}{\partial p'_y} = -x\sin\theta + y\cos\theta \\ z' = \dfrac{\partial S}{\partial p'_z} = z \end{cases}$$

(2.104)

$$\begin{cases} p_x = \dfrac{\partial S}{\partial x} = p'_x\cos\theta - p'_y\sin\theta \\ p_y = \dfrac{\partial S}{\partial y} = p'_x\sin\theta + p'_y\cos\theta \\ p_z = \dfrac{\partial S}{\partial z} = p'_z \end{cases}$$

(2.105)

• 2.9 Trans. can. infinit. et intégrales premières

$$\begin{cases} p'_x = p_x \cos\theta + p_y \sin\theta \\ p'_y = -p_x \sin\theta + p_y \cos\theta \\ p'_z = p_z \end{cases} \quad (2.106)$$

$$S(x, y, z, p'_x, p'_y, p'_z) = (x\cos\theta + y\sin\theta)p'_x + (-x\sin\theta + y\cos\theta)p'_y + zp'_z \quad (2.103)$$

$$S = xp'_x + yp'_y + zp'_z + \theta(yp'_x - xp'_y) \quad (2.107)$$

$$\sigma = \left.\frac{\partial S}{\partial\theta}\right|_{\theta=0} = yp_x - xp_y \quad (2.108)$$

$$[H, \sigma] = 0 \quad \text{et} \quad \frac{d\sigma}{dt} = 0$$

• 2.10 Le théorème de Liouville

$$(d\eta) = dq_1 dq_2 \ldots dq_f dp_1 dp_2 \ldots dp_f$$

$$(d\zeta) = dQ_1 dQ_2 \ldots dQ_f dP_1 dP_2 \ldots dP_f$$

$$(d\zeta) = |dtm(M)|(d\eta)$$ (2.109)

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}$$

$$|dtm(M)|^2 \, dtm(J) = dtm(J)$$ (2.110)

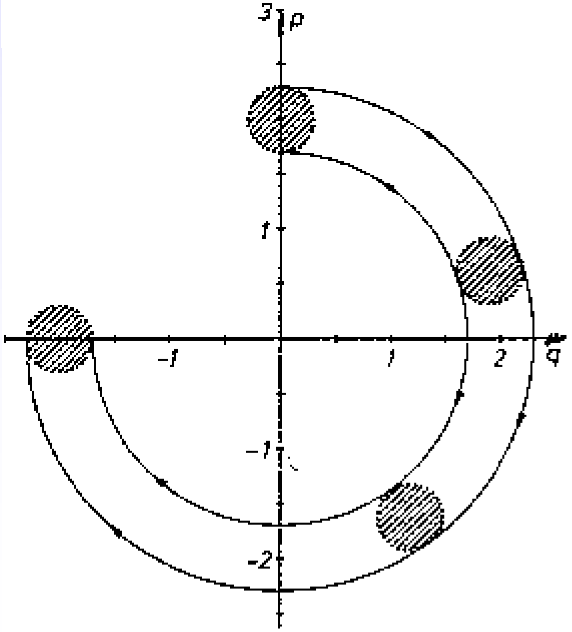
$$(d\zeta) = (d\eta)$$

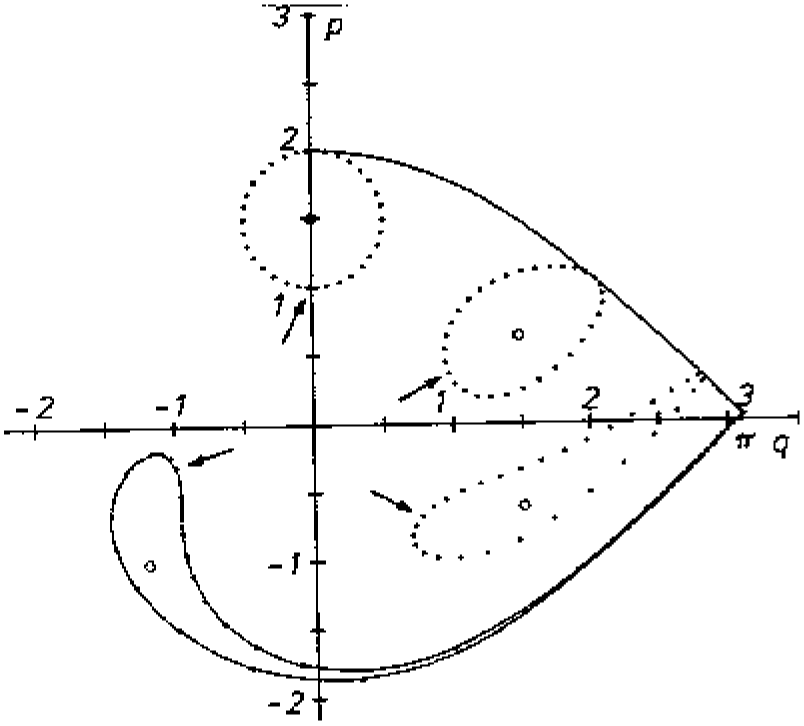
$$\int_\Omega d\eta$$

• 2.10 Le théorème de Liouville

théorème de Liouville :

Les transformations canoniques conservent les volumes dans l'espace de phase





- 2.10 Le théorème de Liouville

$$\rho\left(q,p,t\right)\delta V$$

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+[\rho,H]=0$$