On frequency shifting by elementary modifications of inertia and stiffness

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The problem of shifting eigenvalues presents both a practical interest to avoid resonance and theoretical interest as a computational tool in numerical analysis. In this last context, the general translation of the eigenvalue spectrum is widely used in iteration techniques. Shifting by elementary modifications of the inertia and stiffness matrices opens wider possibilities, especially in the presence of closely spaced eigenvalues or degeneracy. Orthogonal deflation turns out to be one of its particular applications.

1. ELEMENTARY MODIFICATIONS OF INERTIA AND STIFFNESS MATRICES

Consider the eigenvalue problem in n degrees of freedom

$$Kx = \omega^2 Mx \tag{1}$$

where x is the column of modal amplitudes, K and M square symmetric matrices of stiffness and inertia respectively.

By elementary inertia we understand any inertia attached to a given linear form in the generalized coordinates q_i

$$\sum_1^n h_i q_i = h' q$$

where h' denotes the transpose of the column matrix h of the coefficients of the form and q the column matrix of the generalized coordinates. Hence the kinetic energy attached to an elementary inertia μ is of the form

$$\tfrac{1}{2}\mu(h'\dot{q})^2 = \tfrac{1}{2}\dot{q}'(\mu hh')\dot{q}$$

and the addition of this to the already existing kinetic energy

$$\tfrac{1}{2}\dot{q}' M\dot{q}$$

is equivalent to the modification of the inertia matrix M into

$$M + \mu hh'$$

Similarly an additional potential energy

$$\tfrac{1}{2}\kappa(h'q)^2 = \tfrac{1}{2}q'(\kappa hh')q$$

of stiffness coefficient κ, modifies the stiffness matrix into

$$K+\kappa hh'.$$

The changes in design parameters, that produce such elementary modifications of inertia and stiffness, are easily traced on lumped mass models. They are quite difficult to assess on continuous structures, where the reduction to a finite number of degrees of freedom depends on the type of discretization employed. The integral equation method with point collocation is here to be favored, precisely because of its ability to produce a rational lumped mass approximation to the continuum [1].

2. SPECIAL CASES OF ALTERATIONS IN MODE SHAPES AND FREQUENCIES

The modified eigenvalue problem

$$(K+\kappa hh')y = \omega^2(M+\mu hh')y \tag{2}$$

presents some remarkably simple properties for special choices of the elementary modifications.

a. *Invariance of a modal shape and its associated eigenvalue*

If $x_{(m)}$ is a modal shape of problem (1) associated to the eigenvalue ω_m, it remains modal shape of problem (2) without eigenvalue alterations if the coefficients of the linear form are such that $h'x_{(m)} = 0$.

Trivial examples of this property are the addition of a lumped mass or of a spring anchorage, or both, at the level of a node of a flexural vibration mode.

b. *Deflation*

In particular if $h = Mx_{(r)}$, the orthogonality properties of mode shapes

$$x'_{(r)}Mx_{(m)} = 0 \qquad \text{for } r \neq m \tag{3}$$

make all eigensolutions of problem (1) eigensolutions of problem (2) except for one. The exception is $x_{(r)}$ itself that remains a modal shape but whose eigenvalue is shifted to

$$\hat{\omega}_r^2 = \frac{\omega_r^2 + \kappa x'_{(r)} \quad x_{(r)}}{1+\mu x'_{(r)}Mx_{(r)}}. \tag{4}$$

This property of shifting a single eigenvalue in the spectrum implies the knowledge of the associated modal shape. In iteration techniques it is used to force convergence towards a modal shape other than the one $x_{(r)}$ that was just obtained. To this effect the inverse eigenvalue $\lambda_r = 1/\omega_r^2$ is usually shifted to zero by taking

$$\kappa = 0 \qquad \mu = -\frac{1}{x'_{(r)}Mx_{(r)}} \tag{5}$$

Thus the matrix K remains unchanged but M becomes

$$M - \frac{Mx_{(r)}x'_{(r)}M}{x'_{(r)}Mx_{(r)}} = MP_r = P'_rM \tag{6}$$

where $\quad P_r = I - \dfrac{x_{(r)}\{Mx_{(r)}\}'}{x'_{(r)}Mx_{(r)}}$

is an orthogonal projection matrix [2], that clearly does not depend on the undetermined scale of $x_{(r)}$ but only on its 'direction'. The direction of $x_{(r)}$ is that of the projection itself since

$$P_rx_{(r)} = 0;$$

moreover the operator projects in a subspace orthogonal to $x_{(r)}$ since for any vector a

$$x'_{(r)}MP_ra = 0 \qquad \text{i.e.} \qquad x'_{(r)}MP_r = 0.$$

This method of removing $x_{(r)}$ from an iterative convergence process is known as 'orthogonal deflation'.

c. *Elimination of zero frequency modes*

The original matrix K is frequently singular because the system possesses rigid body modes or mechanisms; we refer to both of them as kinematical modes. Let $u_{(i)}$ denote an orthogonalized set of such degrees of freedom, whence

$$Ku_{(i)} = 0 \qquad u'_{(i)}Mu_{(j)} = 0 \qquad i \neq j \tag{7}$$

and they can be considered as modal shapes associated to the zero eigenvalue. In this case the direct iteration procedure of Duncan and Collar [4]

$$K^{-1}Mx = \lambda x \tag{8}$$

fails because the static influence coefficients matrix K^{-1} does not exist. The difficulty can be overcome by a generalization of the notion of influence coefficients [1]. If, instead, we try the idea of eigenvalue shifting through (5), the procedure leads to an indeterminacy of the shifted eigenvalue (4) because in this case $\omega_r^2 = 0$. It is thus necessary to use a non zero κ value. In fact, if we modify the stiffness matrix as follows

$$\hat{K} = K + \sum_i \frac{Mu_{(i)}\{Mu_{(i)}\}'}{u'_{(i)}Mu_{(i)}} = \hat{K}' \tag{9}$$

we have, by virtue of (7)

$$\hat{K}u_{(j)} = Mu_{(j)} \tag{10}$$

and for the modal shapes of non zero frequency, that are orthogonal to all the kinematical modes

$$u'_{(i)}Mx_{(r)} = 0 \qquad \hat{K}x_{(r)} = Kx_{(r)} = \omega_r^2 Mx_{(r)}. \tag{11}$$

Because the $u_{(i)}$ and $x_{(r)}$ constitute a complete orthogonal base for the x vectors, and $Mu_{(i)}$ and $Mx_{(r)}$ are linearly independent, results (10) and (11) show that the new stiffness matrix is non singular and can thus be inverted. The modified iteration scheme

$$\hat{K}^{-1}Mx = \lambda x$$

has the same modal solutions as problem (1), the kinematical modes $u_{(i)}$ being associated to the inverse eigenvalue $\lambda = 1$ in view of (10) and the other modal shapes $x_{(r)}$ keeping their original inverse eigenvalues $\lambda_r = 1/\omega_r^2$ in view of (11). However, the kinematical modes, which are usually known by simple inspection or by solving directly (7), should still be shifted to $\lambda = 0$ in order to force convergence to the other modes. This can be achieved by a modification of M similar to (5)

$$\hat{M} = M - \sum_i \frac{Mu_{(i)}\{Mu_{(i)}\}'}{u'_{(i)}Mu_{(i)}} \tag{12}$$

whereby $\hat{M}u_{(j)} = 0 \qquad \hat{M}x_{(r)} = Mx_{(r)}$. (13)

Thus the low frequency spectrum of inverse eigenvalues associated to the $x_{(r)}$ can be determined from the modified iteration scheme

$$\hat{K}^{-1}\hat{M}x = \lambda x \tag{14}$$

with successive orthogonal deflations.

An alternative is the modified iteration scheme

$$A\hat{K}^{-1}Mx = \lambda x \tag{15}$$

where A is an orthogonal projector [1]

$$A = I - \sum_i \frac{u_{(i)}u'_{(i)}M}{u'_{(i)}Mu_{(i)}} \tag{16}$$

with the properties

$$Au_{(j)} = 0 \qquad Ax_{(r)} = x_{(r)} \tag{17}$$

$\hat{K}^{-1}$ can then be recognized as a matrix of extended influence coefficients [1].

d. *Invariance under any elementary modification*

If we call

$$\theta = (\kappa/\mu)^{\frac{1}{2}} \tag{18}$$

the 'alteration (circular) frequency', any modal shape of problem (1) remains modal shape of problem (2) with the same eigenvalue ω^2, provided $\theta^2 = \omega^2$; this result is independent of the particular linear form involved in the elementary modification of the matrices. The proof is by direct verification.

3. ALTERATIONS OF MODES AND EIGENVALUES IN THE GENERAL CASE

The general solution of the modified problem (2) can be found by expansion of the eigenvectors y in the basis of the eigenvectors of problem (1).

We do not have to distinguish here the kinematical modes, if any, and use the general notation $x_{(r)}$ $(r = 1, 2, \ldots, n)$ for the eigenvectors of problem (1).

$$\text{Setting } y = \sum_1^n \alpha_r x_{(r)}, \tag{19}$$

substituting into (2) and using (11) we find

$$\sum_1^n (\omega_r^2 - \omega^2)\alpha_r M x_{(r)} + \mu(\theta^2 - \omega^2)(\sum_1^n \alpha_r \beta_r) h = 0 \tag{20}$$

$$\text{where } \beta_r = h' x_{(r)}. \tag{21}$$

If this is premultiplied by $x'_{(m)}$ and due account is taken of the orthogonality relations (3), there comes

$$\gamma(\omega_m^2 - \omega^2)\alpha_m + \mu(\theta^2 - \omega^2)(\sum_1^n \alpha_r \beta_r)\beta_m = 0 \quad (m = 1, 2, \ldots, n) \tag{22}$$

This set of linear and homogeneous equations in the unknown coefficients α_m, poses an eigenvalue problem for ω^2 equivalent to (2); it contains the additional assumption that the eigenvectors of problem (1) are normed as follows

$$x'_{(m)} M x_{(m)} = \gamma > 0. \tag{23}$$

Equations (22) are linked to a rather remarkable form of frequency equation; written in the equivalent form

$$\frac{\alpha_m}{\mu(\theta^2 - \omega^2)} + \frac{\beta_m}{\gamma(\omega_m^2 - \omega^2)}(\Sigma \alpha_r \beta_r) = 0 \tag{24}$$

multiplied by their respective coefficient β_m and added, they yield after cancellation of the common factor $(\sum \alpha_r \beta_r)$

$$\phi(\omega^2) = \frac{1}{\mu(\theta^2 - \omega^2)} + \sum_1^n \frac{\beta_m^2}{\gamma(\omega_m^2 - \omega^2)} = 0. \tag{25}$$

For $\mu > 0$, $\kappa > 0$ and consequently θ real, ϕ is a monotonic increasing function of its argument ω^2. It jumps from plus to minus infinity each time it overtakes an eigenvalue ω_m^2 of problem (1) (provided $\beta_m \neq 0$) and also when it overtakes the value θ^2. Its zeros are the new eigenvalues. Consequently each original frequency is normally shifted towards the alteration frequency but keeps its rank in the frequency spectrum.

The exceptional cases, like the vanishing of a β_m or the coalescence of the alteration frequency with some original frequency, were dealt with in section 2. If the modifications brought to M and K are small enough to be treated as perturbations, explicit forms can be given to the shifts in eigenvalues

$$\hat{\omega}_r^2 = \frac{\gamma\omega_r^2 + \kappa\omega_r^2}{\gamma + \mu\beta_r^2} \tag{26}$$

and to the new modal shapes

$$y_{(r)} = x_{(r)} + \sum_{m \neq r} \alpha_m x_{(m)} \tag{27}$$

$$\alpha_m = -\frac{\mu(\theta^2 - \hat{\omega}_r^2)\beta_r\beta_m}{\gamma(\omega_r^2 - \hat{\omega}_r^2)} \tag{28}$$

4. DEGENERACY

For simplicity take first a case of simple degeneracy: two linearly independent mode shapes $x_{(m1)}$ and $x_{(m2)}$ belong to the same eigenvalue ω_m^2 in problem 1. There always exists some non trivial linear combination

$$a_m = \varepsilon_1 x_{(m1)} + \varepsilon_2 x_{(m2)}$$

of the mode shapes, such that for an arbitrarily given h

$$h' a_m = 0.$$

Indeed the linear homogeneous equation in ε_1 and ε_2

$$\{h' x_{(m1)}\}\varepsilon_1 + \{h' x_{(m2)}\}\varepsilon_2 = 0$$

has always at least one non trivial solution. But then, as shown in section 2. a, a_m which is a modal shape of eigenvalue ω_m^2 of problem (1) remains an eigensolution of problem (2). The generalization is obvious, if problem (1) has an index of degeneracy p for some ω_m^2, $q < p$ elementary modifications of the matrices M and K with arbitrary coefficients, will leave at least $p-q$ eigensolutions invariant.

5. CLOSELY SPACED EIGENVALUES

For exact degeneracy the iteration procedure involves theoretically no difficulties. Convergence must occur to some modal shape of the linear manifold belonging to the degenerate eigenvalue. Subsequent deflation will force convergence to an orthogonal mode shape of the same manifold and the procedure may in principle be repeated until this manifold has been completely spanned. In practice however, round-off errors will tend to replace degeneracy by close spacing of eigenvalues. It is also true that the problem of determining modes and frequencies of modern complex aerospace structures almost invariably leads to the existence of quasi-degeneracy. It may be suspected that in this case the use to which this frequency analysis is directed does not necessarily warrant a very accurate discrimination between closely spaced modal shapes. Nevertheless the problem is an irritating one because of the apparent slowness of convergence and lack of accuracy criteria.

We will restrict our analysis to the case of two closely spaced modal shapes a_1 and a_2 satisfying

$$Ma_1 = \lambda_1 Kx_1 \qquad Ma_2 = \lambda_2 Kx_2 \tag{29}$$

with small difference between the inverse eigenvalues λ_1 and λ_2.

After a sufficient number of iterations, assuming all modal shapes of higher λ to have been deflated, we can obtain an iterate x that is, with sufficient accuracy, a linear combination

$$x = \alpha_1 a_1 + \alpha_2 a_2 \tag{30}$$

of the unknown modal shapes. Without loss of generality it will also be assumed that we have the norms

$$a_1' Ma_1 = 1 \qquad a_2' Ma_2 = 1 \qquad x' Mx = 1. \tag{31}$$

The next iterates will be

$$y = K^{-1} Mx = \alpha_1 \lambda_1 a_1 + \alpha_2 \lambda_2 a_2 \tag{32}$$

$$z = K^{-1} My = \alpha_1 \lambda_1^2 a_1 + \alpha_2 \lambda_2^2 a_2. \tag{33}$$

If the convergence process becomes very slow, we are certain that both α_1 and α_2 are different from zero. Then, setting up the Rayleigh quotient

$$\lambda = \frac{(z+\varepsilon y)' M(z+\varepsilon y)}{(z+\varepsilon y)' K(z+\varepsilon y)} = \frac{(z+\varepsilon y)' M(z+\varepsilon y)}{(z+\varepsilon y)' M(y+\varepsilon x)} \tag{34}$$

it is clear that its extremal values are respectively λ_1 and λ_2.

Indeed the extremizing condition turns out to be a second degree algebraic equation for ε, whose roots will furnish respectively a_1 and a_2 and the associated eigenvalues. Unfortunately, because (32) and (33) have not very different shapes from (30), this approach is known to be unstable by ill-conditioning. This is not too surprising as we would have practically determined two modal shapes through a single iteration sequence.

Let us turn to the possibilities offered by frequency shifting and modify the mass matrix into

$$\hat{M} = M - \frac{Mxx'M}{x'Mx} \tag{35}$$

whereby our x becomes a modal shape associated to $\lambda = 0$. Since the mode shapes of lower inverse eigenvalues than λ_1 and λ_2 remain eigensolutions of the modified problem, a new iteration cycle will hopefully provide an iterate y, quite different from x, that is a new linear combination of the exact modes

$$y = \beta_1 a_1 + \beta_2 a_2. \tag{36}$$

To this effect we must show that the inverse eigenvalue associated to y remains higher than those of the next modes. In view of the results obtained in the analysis of degeneracy, it may indeed be suspected that in

$$\hat{M}y = \lambda K y \tag{37}$$

the value of λ will lie between λ_1 and λ_2. To prove this we observe that (31) is equivalent to

$$\alpha_1^2 + \alpha_2^2 = 1.$$

If furthermore we norm y

$$y'My = 1 \quad \text{equivalent to} \quad \beta_1^2 + \beta_2^2 = 1$$

and $x'My = \alpha_1 \beta_1 + \alpha_2 \beta_2$.

Consequently we can write

$$\alpha_1 = \cos\phi \qquad \alpha_2 = \sin\phi \qquad \beta_1 = \cos\theta \qquad \beta_2 = \sin\theta$$
$$x'My = \cos(\phi - \theta).$$

From (35) we find then

$$\hat{M}y = \sin\phi \sin(\phi - \theta) M a_1 + \cos\phi \sin(\theta - \phi) M a_2$$

and, in view of (29)

$$\hat{M}y = \sin(\phi - \theta)\{\sin\phi \lambda_1 K a_1 - \cos\phi \lambda_2 K a_2\}.$$

Equation (37) is then split into

$$\lambda_1 \sin\phi \sin(\phi - \theta) = \lambda \cos\theta \tag{38}$$
$$\lambda_2 \cos\phi \sin(\phi - \theta) = -\lambda \sin\theta. \tag{39}$$

Elimination of θ between those equations yields two solutions for λ.

One is $\lambda = 0$, requiring $\theta = \phi$, and confirming the shifting of x to zero λ, the other is

$$\lambda = \lambda_1 \sin^2\phi + \lambda_2 \cos^2\phi \tag{40}$$

which lies, as was hoped, between λ_1 and λ_2.

Furthermore, eliminating λ between (38) and (39), the result

$$\lambda_1 \sin\theta \sin\phi + \lambda_2 \cos\theta \cos\phi = 0 \tag{41}$$

shows that

$$x'My = \cos(\phi-\theta) = \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\cos(\phi+\theta) = \varepsilon \tag{42}$$

is small, being a measure of the relative gap between the eigenvalues; hence y is almost orthogonal to x. We find then that

$$\hat{M}y = My-\varepsilon Mx = \lambda Ky$$

from which the useful results

$$x'Ky = 0 \tag{43}$$

$$1-\varepsilon^2 = \lambda y'Ky \tag{44}$$

We are now able to determine the modes a_1 and a_2 and their eigenvalues from the linear combinations $x+\gamma y$ that extremize the quotient

$$\frac{(x+\gamma y)'M(x+\gamma y)}{(x+\gamma y)'K(x+\gamma y)} = \frac{1+2\gamma\varepsilon+\gamma^2}{\dfrac{1}{\eta}+\gamma^2\dfrac{1-\varepsilon^2}{\lambda}} \tag{45}$$

where $$\eta = \frac{x'Mx}{x'Kx} = \frac{1}{x'Kx}. \tag{46}$$

The extremizing condition is found to be

$$\gamma^2 - \frac{\mu-1}{\varepsilon}\gamma - \mu = 0 \qquad \text{where} \qquad \mu = \frac{\lambda}{\eta(1-\varepsilon^2)} \tag{47}$$

The accuracy with which the roots of this equation can be determined still depends on the accuracy of the ratio of the two small numbers $\mu-1$ and ε. This accuracy is however obtainable because of the near orthogonality of the mode shapes x and y. The same basic idea can obviously be applied to the case where several eigenvalues are nearly coincident.

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