

Deciding game invariance

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Abstract

In [12], Duchêne and Rigo introduced the notion of invariance for take-away games on heaps. Roughly speaking, these are games whose rulesets do not depend on the position. Given a sequence S of positive tuples of integers, the question of whether there exists an invariant game having S as set of \mathcal{P} -positions is relevant. In particular, it was recently proved by Larsson et al. [23] that if S is a pair of complementary Beatty sequences, then the answer to this question is always positive. In this paper, we show that for a fairly large set of sequences (expressed by infinite words), the answer to this question is decidable.

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1. Introduction

Let $n \geq 1$ be an integer. In this paper, we consider take-away impartial games played over n piles of tokens. Two players alternatively remove a positive number of tokens from one or several piles following a prescribed ruleset. The rules are the same for both players. We assume normal convention, i.e., the player making the last move wins. Since we always remove a positive number of tokens, the game is acyclic and there is always a winner.

A *position* of such a game is an n -tuple of non-negative integers which corresponds to the number of tokens available in each pile. A *move* is also an n -tuple of non-negative integers corresponding to the number of tokens that are removed from each pile. Let $\mathbf{p} = (p_1, \dots, p_n)$ be a position and $\mathbf{m} = (m_1, \dots, m_n)$ be a non-zero move. The move \mathbf{m} can be applied to the position \mathbf{p} provided that

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$\mathbf{m} \leq \mathbf{p}$, i.e., for all i , $m_i \leq p_i$. The position resulting of the application of \mathbf{m} is the n -tuple $\mathbf{p} - \mathbf{m}$.

Definition 1. A game, played over n piles, is given by a function $G : \mathbb{N}^n \rightarrow 2^{\mathbb{N}^n}$ that maps every position \mathbf{p} to a set of moves that can be chosen from \mathbf{p} by the player. Otherwise stated, the ruleset is provided by the map G . For a position \mathbf{p} , the set of options of \mathbf{p} is the set $\{\mathbf{p} - \mathbf{m} \mid \mathbf{m} \in G(\mathbf{p})\}$ of positions where the player can move directly. A strategy consists in choosing a particular option for every position.

An interval of integers is denoted by $\llbracket k, \ell \rrbracket$. For an example of take-away game, the game of Nim over 2 piles is described by the map

$$G_{\text{NIM}} : \mathbb{N}^2 \rightarrow 2^{\mathbb{N}^2}, (x, y) \mapsto \{(i, 0) \mid i \in \llbracket 1, x \rrbracket\} \cup \{(0, j) \mid j \in \llbracket 1, y \rrbracket\}.$$

For Wythoff's game, the description is given by

$$G_{\text{WYTHOFF}} : \mathbb{N}^2 \rightarrow 2^{\mathbb{N}^2}, (x, y) \mapsto G_{\text{NIM}}(x, y) \cup \{(k, k) \mid k \in \llbracket 1, \min\{x, y\} \rrbracket\}.$$

With such a formal presentation, we recall the notion of invariant game introduced in [12]. Note that we shall later on distinguish two notions of invariance: invariant games and admissible subsets.

Definition 2. A game $G : \mathbb{N}^n \rightarrow 2^{\mathbb{N}^n}$ is invariant if there exists a set $I \subseteq \mathbb{N}^n$ such that, for all positions \mathbf{p} , we have

$$G(\mathbf{p}) = I \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \leq \mathbf{p}\}.$$

Otherwise stated, we may apply exactly the same moves to every position, with the only restriction that there are enough tokens left. Since a game is defined by its moves, formally by the map G , one also speaks of invariant moves.

A motivation to introduce the notion of invariance is the relative simplicity of the corresponding rulesets. Roughly speaking, one has “just” to remember the set I .

The game of Nim defined above is invariant. Simply consider the set

$$I_{\text{NIM}} = \{(i, 0) \mid i \geq 1\} \cup \{(0, j) \mid j \geq 1\}.$$

Similarly, Wythoff's game is invariant with the set

$$I_{\text{WYTHOFF}} = I_{\text{NIM}} \cup \{(k, k) \mid k \geq 1\}.$$

For an example of non-invariant game, consider the following map,

$$G_{\text{EVEN}} : \mathbb{N}^2 \rightarrow 2^{\mathbb{N}^2}, (x, y) \mapsto \begin{cases} \{(i, 0) \mid i \in \llbracket 1, x \rrbracket\}, & \text{if } x + y \text{ is even;} \\ \{(i, i) \mid i \in \llbracket 1, \min\{x, y\} \rrbracket\}, & \text{otherwise.} \end{cases}$$

Here, the moves that can be applied from a position (x, y) depend on the position itself.

Recently, Fraenkel and Larsson introduced a generalization of this notion of invariance [18].

Definition 3. Let $t \geq 1$ be an integer. A game $G : \mathbb{N}^n \rightarrow 2^{\mathbb{N}^n}$ is t -invariant if the set of positions can be partitioned into t subsets S_1, \dots, S_t and there exist t sets $I_1, \dots, I_t \subseteq \mathbb{N}^n$ such that, for all positions \mathbf{p} ,

$$\text{if } \mathbf{p} \in S_j, \text{ then } G(\mathbf{p}) = I_j \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \leq \mathbf{p}\}.$$

In particular, an invariant game is 1-invariant.

Example 4. The game G_{EVEN} is clearly 2-invariant. One considers the partition of \mathbb{N}^2 into $S_1 = \{(x, y) \mid x + y \text{ is even}\}$ and $S_2 = \{(x, y) \mid x + y \text{ is odd}\}$.

Note that there exist some games which are not t -invariant for any t .

Example 5. The game

$$G_{\text{MARK}} : \mathbb{N} \rightarrow 2^{\mathbb{N}}, x \mapsto \{1, \lceil x/2 \rceil\}$$

defined in [15] is not t -invariant for any t .

It is classical to associate a set of \mathcal{P} -positions with a game.

Definition 6. A position $\mathbf{p} \in \mathbb{N}^n$ is a \mathcal{P} -position if there exists a strategy for the second player (i.e., the player who will play on the next round) to win the game, whatever the move of the first player is. We let $\mathcal{P}(G)$ denote the set of \mathcal{P} -positions of the game G . Conversely, \mathbf{p} is an \mathcal{N} -position if there exists a winning strategy for the first player (i.e., the one who is making the current move).

The characterization of the set of \mathcal{P} -positions of an impartial acyclic game is well-known.

Proposition 7. The sets of \mathcal{P} - and \mathcal{N} -positions of an impartial acyclic game are uniquely determined by the following two properties:

- Every move from a \mathcal{P} -position leads to an \mathcal{N} -position (stability property of the set of \mathcal{P} -positions).
- From every \mathcal{N} -position, there exists a move leading to a \mathcal{P} -position (absorbing property of the set of \mathcal{P} -positions).

Remark 8. Two different games G and H can be such that $\mathcal{P}(G) = \mathcal{P}(H)$. For an example, the game

$$H : \mathbb{N}^2 \rightarrow 2^{\mathbb{N}^2},$$

$$(x, y) \mapsto G_{\text{WYTHOFF}}(x, y) \cup (\{(3, 1), (5, 1), (6, 1)\} \cap \{(i, j) \mid i \leq x, j \leq y\})$$

is such that $\mathcal{P}(G_{\text{WYTHOFF}}) = \mathcal{P}(H)$. See [9] for details about the moves that can be adjoined to Wythoff's game without modifying the set of \mathcal{P} -positions. Here G_{WYTHOFF} and H are both invariant games, but one can also imagine variant games leading to the same set of \mathcal{P} -positions such as

$$H' : \mathbb{N}^2 \rightarrow 2^{\mathbb{N}^2},$$

$$(x, y) \mapsto \begin{cases} G_{\text{WYTHOFF}}(x, y) \cup \{(5, 1), (6, 1)\} \cap \{(i, j) \mid i \leq x, j \leq y\}, & \text{if } x \text{ even;} \\ G_{\text{WYTHOFF}}(x, y) \cup \{(1, 3), (1, 5)\} \cap \{(i, j) \mid i \leq x, j \leq y\}, & \text{if } x \text{ odd.} \end{cases}$$

Again, one can check that $\mathcal{P}(G_{\text{WYTHOFF}}) = \mathcal{P}(H')$. A few extra moves are adjoined to the usual Wythoff moves. These moves do not modify the set of \mathcal{P} -positions. But observe that the added moves depend on the parity of x , thus this game is not invariant. For other variant games having $\mathcal{P}(G_{\text{WYTHOFF}})$ as set of \mathcal{P} -positions, see [1]. Observe that H' is an example of 2-invariant game.

This leads to the following definition.

Definition 9. A subset X of \mathbb{N}^n is t -admissible if there exists a t -invariant game G such that $X = \mathcal{P}(G)$.

In this paper, we deal with the question of t -admissibility of subsets of \mathbb{N}^n . We first show that every subset of \mathbb{N}^n , which contains $\mathbf{0} = (0, \dots, 0)$, is 2-admissible (Theorem 11). Thus the general question addressed in this paper is the following one. Given a subset X of \mathbb{N}^n , is X 1-admissible? In [23], the authors proved that if X is a pair of complementary homogeneous Beatty sequences, then X is 1-admissible. Recently, the case of non-homogeneous Beatty sequences was investigated [7]: a partial characterization is given for sets which are 1-admissible. In the current paper, using the formalism of first-order logic, we show that for a wide range of sets (not only Beatty sequences), the problem turns out to be decidable. More precisely, this problem is decidable for sets that are definable in the Presburger arithmetic $\langle \mathbb{N}, + \rangle$ extended with a unary map V_U related to expansions of integers in a numeration system. The precise framework is given in Section 3. As a particular case, we will also consider sets of \mathcal{P} -positions of existing variant games, such as the rat game and the mouse game [16], the Tribonacci game [10], Pisot cubic games [11], Mark [15], etc.

As an example, consider the Tribonacci game $G_{\text{TRIBO}} : \mathbb{N}^3 \rightarrow 2^{\mathbb{N}^3}$, played over three piles of tokens, where the variant rules are described in [10]. Its set of \mathcal{P} -positions is coded by the Tribonacci word T

$$T = 12131211213121213121121312131211213121213121 \dots, \quad (1)$$

which is the unique fixed point of the morphism over $\{1, 2, 3\}^*$ given by $f : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$. If letters of T are indexed by positive integers, the m th \mathcal{P} -position is given by the index of the m th occurrence of the letter 1, 2 and 3 respectively. The first few \mathcal{P} -positions in $\mathcal{P}(G_{\text{TRIBO}})$ are

$$(1, 2, 4), (3, 6, 11), (5, 9, 17), (7, 13, 24), (8, 15, 28), (10, 19, 35), (12, 22, 41), \dots$$

Up to implementation, our main result shows that it is decidable whether $\mathcal{P}(G_{\text{TRIBO}})$ is 1-admissible. For an introduction to combinatorics on (infinite) words, see, for instance, [2, 25]. In this paper, we only use a few properties and definitions about words. For the Tribonacci word, one has to consider the sequence of finite words $(f^n(1))_{n \geq 1}$ which converges to T . We also assume that

the reader has some knowledge about automata theory. See for instance [30]. Indeed, in Section 5, we recall that the first-order logical formalism is equivalent to a representation in terms of languages accepted by finite automata.

Remark 10. *Contrasting with our result, some other questions on invariant games lead to undecidable problems. It is shown in [24] that deciding whether two 1-invariant games have the same sets of \mathcal{P} positions is undecidable.*

This paper is organized as follows. In Section 2, we quickly show that every subset of \mathbb{N}^n containing $\mathbf{0} = (0, \dots, 0)$ is 2-admissible. In Section 3, we describe the formalism of Pisot numeration systems and the corresponding first-order definable sets. We explain in Section 4 how this formalism leads to a decision procedure about the 1- or 2-admissibility of subsets of \mathbb{N}^n . Next, in Section 5, we reformulate our result in terms of sets recognizable by means of finite automata. Section 6 is dedicated to applications of this procedure. We consider several classical games. We can decide the 1-admissibility of the set of \mathcal{P} -positions of 2-heap games defined in [14], (generalized) Tribonacci game [10, 11], Raleigh game [17], games coded by periodic words or by Parikh-constant morphic words. Note that in the periodic case, we also provide an independent algorithm.

2. Admissible set of positions

If A, B are subsets of \mathbb{N}^n , $A + B$ (resp., $A - B$) denotes the set $\{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$ (resp., $\{\mathbf{a} - \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$).

Theorem 11. *Every subset P of \mathbb{N}^n which contains $\mathbf{0} = (0, \dots, 0)$ is 2-admissible.*

PROOF. We define a partition of \mathbb{N}^n into two subsets S_1 and S_2 as follows. First we make use of an auxiliary set S . Let \mathbf{x} be not in P . If there exists $\mathbf{y} \in P$ such that $\mathbf{y} \leq \mathbf{x}$ and $\mathbf{x} - \mathbf{y} \notin P - P$, then \mathbf{x} belongs to S . Otherwise, if, for all \mathbf{y} in P such that $\mathbf{y} \leq \mathbf{x}$, we have $\mathbf{x} - \mathbf{y} \in P - P$ then \mathbf{x} belongs to S_2 . To get a partition of \mathbb{N}^n , we define S_1 as $S \cup P$.

We define $I_1 = \mathbb{N}^n \setminus (P - P)$ and $I_2 = S_2$. The game $G : \mathbb{N}^n \rightarrow 2^{\mathbb{N}^n}$ defined, for every $\mathbf{p} \in \mathbb{N}^n$, by

$$\text{if } \mathbf{p} \in S_j, \text{ then } G(\mathbf{p}) = I_j \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \leq \mathbf{p}\}, \text{ for } j = 1, 2,$$

is 2-invariant. Clearly, this game has P as set of \mathcal{P} -positions, i.e., $\mathcal{P}(G) = P$. Indeed, with this definition, a player in a position $\mathbf{p} \in P$ can only play to a position not in P . Now assume that we have a position $\mathbf{p} \notin P$. If \mathbf{p} is in S_1 then, by definition of S , there is an option of \mathbf{p} in P . If \mathbf{p} is in S_2 , then the move \mathbf{p} leading directly to $\mathbf{0} \in P$ is allowed.

In addition, the question of the 1-admissibility of a given set P can be reduced as whether a particular 1-invariant game admits P as set of \mathcal{P} -positions. To find this game, let us make a preliminary observation. A move can be adjoined to an impartial acyclic game without changing the set of \mathcal{P} -positions

if and only if it does not alter the stability property (defined in Proposition 7). Indeed, adding a move leading from one \mathcal{P} -position to another \mathcal{P} -position would necessarily change the stability property of the \mathcal{P} -positions (by Proposition 7). On the other hand, adding a move which does not correspond to a move between any two \mathcal{P} -positions means that both properties of Proposition 7 remain true. Therefore, a move \mathbf{m} can be added if and only if it does not correspond to a move from a \mathcal{P} -position to another \mathcal{P} -position.

Lemma 12. *A subset P of \mathbb{N}^n is 1-admissible if and only if the 1-invariant game H defined by*

$$H(\mathbf{p}) = (\mathbb{N}^n \setminus (P - P)) \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \leq \mathbf{p}\}, \text{ for every } \mathbf{p},$$

has P as set of \mathcal{P} -positions, i.e., $P = \mathcal{P}(H)$.

PROOF. Assume that there exists a 1-invariant game G such that $P = \mathcal{P}(G)$, i.e., there exists a subset I such that $G(\mathbf{p}) = I \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \leq \mathbf{p}\}$ for every \mathbf{p} . For all positions \mathbf{p} , we have

$$I \subseteq (\mathbb{N}^n \setminus (P - P)), \text{ and thus } G(\mathbf{p}) \subseteq H(\mathbf{p}),$$

because if a move \mathbf{m} belongs to I , then \mathbf{m} cannot be written as $\mathbf{x} - \mathbf{y}$ for two distinct elements $\mathbf{x}, \mathbf{y} \in P$. Indeed if that were the case, $\mathbf{m} = \mathbf{x} - \mathbf{y}$ would belong to $G(\mathbf{x})$, and we would be able to play from $\mathbf{x} \in P$ to another element $\mathbf{y} \in P$. This contradicts the fact that P is the set of \mathcal{P} -positions of G .

If we compare the two games, H is an extension of G : for every position, we could have more options in H than in G . Nevertheless, for a position in P if there are more options available in H , the new options do not belong to P . Using Proposition 7, we deduce that $P = \mathcal{P}(G) = \mathcal{P}(H)$.

3. A first-order logic formalism

We start with a minimal background on (Pisot) numeration systems, and then define what is a U -definable subset of \mathbb{N}^n . We conclude this section by the statement of Büchi's theorem which is at the center of our decision procedure.

A real number $\alpha > 1$ is a *Pisot number* if it is the root of polynomial P over \mathbb{Z} whose dominant coefficient is 1 (α is an algebraic integer) and whose all the others roots have modulus less than one.

Definition 13. [3] *A Pisot numeration system is an increasing sequence $(U_i)_{i \geq 0}$ of integers such that $U_0 = 1$ and $(U_i)_{i \geq 0}$ satisfies a linear recurrence relation whose characteristic polynomial is the minimal polynomial of a Pisot number α .*

If $U = (U_i)_{i \geq 0}$ is a Pisot numeration system, every non-negative integer m has a unique U -expansion denoted by $\text{rep}_U(m) = c_\ell \cdots c_0$ computed by a greedy algorithm [13]. It satisfies, for all $r \leq \ell$,

$$\sum_{j=0}^r c_j U_j < U_{r+1}$$

and $c_\ell \neq 0$. The Pisot condition implies that there exists $d > 0$ such that $\lim_{i \rightarrow +\infty} U_i/\alpha^i = d$. Hence U_{i+1}/U_i converges to α , and the digit-set for U -expansions is finite and equal to $\llbracket 0, \sup[U_{i+1}/U_i] - 1 \rrbracket$.

Example 14. Let $k \geq 2$ be an integer. The sequence $(k^i)_{i \geq 0}$ is a Pisot numeration system. It is the usual base- k number system. As a special case, we let $\text{rep}_k(m)$ denote the usual base- k expansion of m with digit-set $\{0, \dots, k-1\}$.

Example 15. Let $(F_i)_{i \geq 0}$ be the Fibonacci sequence defined by $F_0 = 1$, $F_1 = 2$ and $F_{i+2} = F_{i+1} + F_i$ for all $i \geq 0$. The characteristic polynomial of $(F_i)_{i \geq 0}$ is $X^2 - X - 1$. It is the minimal polynomial of $\tau = \frac{1+\sqrt{5}}{2}$, which is a Pisot number. The F -expansions are words over the digit-set $\{0, 1\}$ which do not contain the factor 11. The F -expansions of the first few positive integers are

1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, 10010, 10100, 10101, 100000, \dots

Example 16. Let $(T_i)_{i \geq 0}$ be the sequence defined by $T_0 = 1$, $T_1 = 2$, $T_2 = 4$ and $T_{i+3} = T_{i+2} + T_{i+1} + T_i$ for all $i \geq 0$. The characteristic polynomial of $(T_i)_{i \geq 0}$ is $X^3 - X^2 - X - 1$ which is the minimal polynomial of a Pisot number. This system is referred to as the Tribonacci numeration system. Again the digit-set is $\{0, 1\}$, and T -expansions are words avoiding the factor 111. The T -expansions of the first few positive integers are given in Table 1

	13	7	4	2	1		24	13	7	4	2	1
1					1	14	1	0	0	0	1	
2				1	0	15	1	0	0	1	0	
3				1	1	16	1	0	0	1	1	
4			1	0	0	17	1	0	1	0	0	
5			1	0	1	18	1	0	1	0	1	
6			1	1	0	19	1	0	1	1	0	
7		1	0	0	0	20	1	1	0	0	0	
8		1	0	0	1	21	1	1	0	0	1	
9		1	0	1	0	22	1	1	0	1	0	
10		1	0	1	1	23	1	1	0	1	1	
11		1	1	0	0	24	1	0	0	0	0	0
12		1	1	0	1	25	1	0	0	0	0	1
13	1	0	0	0	0	26	1	0	0	0	1	0

Table 1: The T -representations of the first few integers.

We recall basic definition about formal logic. See, for instance, [3] or [29, Chapter 6]. Let U be a Pisot numeration system. The alphabet A_U of our first-order language contains countably many variables x_1, x_2, x_3, \dots (or x, y, z, \dots) and extra symbols $+, =, \forall_U, \forall, \wedge, \rightarrow, \leftrightarrow, \neg, \exists, \forall$ as well as parentheses. We define terms and formulae inductively and we often make use of extra parentheses for the sake of clarity of the constructions.

Definition 17. First we define terms which are particular words over A_U^* . To construct them, we may apply the following rules finitely many times.

- Any variable is a term.
- If t is a term, then $V_U(t)$ is a term.
- If t_1, t_2 are terms, then $(t_1 + t_2)$ is a term.

Second we define formulae inductively by applying the following rules finitely many times. Note that these formulae are also words over A_U^* .

- If t_1, t_2 are terms, then $t_1 = t_2$ is a formula.
- If φ and ψ are formulae, then $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, $\neg\varphi$ are formulae.
- If φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulae.

The set of formulae over A_U is the first-order language that we shall consider. We denote by $\mathfrak{n}_U = \langle \mathbb{N}, +, V_U \rangle$ this structure.

We turn to the interpretation (i.e., semantics) of these formulae. Variables are ranging over \mathbb{N} . Note that we can only quantify over variables (in contrast with second-order logic). The function V_U maps $m > 0$ to the smallest U_i appearing in the U -expansion of m with a non-zero coefficient. We set $V_U(0) = 1$. In the special case of an integer base system, $V_k(m)$ is the largest power of k dividing m . The other symbols carry their usual interpretation.

Let φ be a *sentence*, i.e., a formula with no free variable (all the variables are under the scope of a quantifier). We write $\mathfrak{n}_U \models \varphi$ if the formula φ is satisfied under the usual interpretation of the symbols. As an example, we write

$$\mathfrak{n}_U \models (\forall x)(\exists y)(x = y + y \vee x = y + y + 1)$$

to express that every non-negative integer x is either even or odd (adding one is permitted, see Remark 19). The set of sentences is called the *first-order theory* (of the corresponding language).

Now assume that φ is a formula where n free variables x_1, \dots, x_n occur. We write $\varphi(x_1, \dots, x_n)$ to highlight the presence of these free variables. If we substitute all the occurrences of x_1, \dots, x_n with constants $d_1, \dots, d_n \in \mathbb{N}$ respectively, then the resulting formula is a sentence and either $\mathfrak{n}_U \models \varphi(d_1, \dots, d_n)$ or not. Given a formula $\varphi(x_1, \dots, x_n)$, we can therefore consider the n -tuples $(d_1, \dots, d_n) \in \mathbb{N}^n$ such that $\mathfrak{n}_U \models \varphi(d_1, \dots, d_n)$. This leads to the following definition.

Definition 18. Let $n \geq 1$. Let $U = (U_i)_{i \geq 0}$ be a Pisot numeration system. A set $X \subseteq \mathbb{N}^n$ is U -definable if there exists a formula $\varphi(x_1, \dots, x_n)$ in $\langle \mathbb{N}, +, V_U \rangle$ such that

$$X = \{(d_1, \dots, d_n) \in \mathbb{N}^n \mid \mathfrak{n}_U \models \varphi(d_1, \dots, d_n)\}.$$

If U is the usual base- k number system, we speak of k -definable sets.

Remark 19. We can define integer constants, inequality relation, multiplication by a constant, and Euclidean division by a constant within $\langle \mathbb{N}, +, V_U \rangle$. The order relation $x \leq y$ is defined by $(\exists z)(y = x + z)$, as well as $x < y$ defined by $(x \leq y) \wedge \neg(x = y)$. The constant 0 is defined by $x = 0 \equiv (\forall y)(x \leq y)$. The successor function S mapping n to $n + 1$ is defined by $S(x) = y \equiv (x < y) \wedge (\forall z)((x < z) \rightarrow (y \leq z))$. To define $r \cdot x$, where r is a constant, one has to write $x + \dots + x$, where the sum has r terms.

Example 20. Consider the formula

$$\psi(x) \equiv (\exists y)(x = y + y).$$

The formula ψ defines the set of even integers (whatever is the Pisot numeration system U). Let $k \geq 2$. The formula

$$\nu(x) \equiv V_k(x) = x$$

defines the set of powers of k . We can say that this set is k -definable. Let $(T_i)_{i \geq 0}$ be the Tribonacci sequence from Example 16. The formula

$$\phi(n) \equiv (\exists x)(\exists y)(x \neq y \wedge V_T(x) = x \wedge V_T(y) = y \wedge n = x + y)$$

defines the set of number whose T -expansion contains exactly two non-zero terms. The first few integers in the set defined by ϕ are 3, 5, 6, 8, 9, 11, 14, 15 because their T -expansions contain exactly two symbols 1.

Given any sentence φ in $\langle \mathbb{N}, +, V_U \rangle$, there exists an algorithm to check whether $n_U \models \varphi$ holds. This result is known as Büchi's theorem. See [5, 4].

Theorem 21. The first-order theory $\langle \mathbb{N}, +, V_U \rangle$ is effectively decidable.

Remark 22. This result has been recently used to get positive results in combinatorics on words. Implementations to deal with the Fibonacci and Tribonacci numerations systems have been developed [8, 27]. With these implementations (mostly relying on automata recognizing addition in these systems) many properties of the Fibonacci, Tribonacci and related infinite words are proved automatically on a laptop with computing time ranging from a few seconds to two hours. The source code developed by the authors is available in [26]. Also see [22] for an example about integer base systems.

4. Decision procedure

We have now collected enough material to settle the decidability result of this paper:

Theorem 23. Let $n \geq 1$. Let U be a Pisot numeration system. Let P be a U -definable subset of \mathbb{N}^n containing $\mathbf{0}$. It is decidable whether P is an 1-admissible set.

PROOF. Without loss of generality, we may assume that $n = 2$. Let $\pi(x, y)$ be a first-order formula defining P . The set $(P - P) \cap \mathbb{N}^2$ is U -definable by the following formula

$$\varphi(x, y) \equiv (\exists i_1)(\exists i_2)(\exists j_1)(\exists j_2)(\pi(i_1, i_2) \wedge \pi(j_1, j_2) \wedge j_1 \geq i_1 \wedge j_2 \geq i_2 \wedge x = j_1 - i_1 \wedge y = j_2 - i_2).$$

From the above lemma, the fact that P is 1-admissible can be expressed by the validity of the sentence

$$\nu \equiv (\forall p_1)(\forall p_2)(\neg \pi(p_1, p_2) \rightarrow (\exists x)(\exists y)(\pi(x, y) \wedge x \leq p_1 \wedge y \leq p_2 \wedge \neg \varphi(p_1 - x, p_2 - y))).$$

Indeed, for every position (p_1, p_2) which is not in P , we are looking for $(x, y) \in P$ and a move not in $P - P$ from (p_1, p_2) to (x, y) . If this holds for every (p_1, p_2) not in P , then the invariant game defined in Lemma 12 has P as set of \mathcal{P} -position. We finish the proof by an application of Theorem 21. We can decide if the sentence ν holds.

5. Recognizable sets and synchronized sequences

A key ingredient for the proof of Theorem 21 comes from automata theory. Büchi's proof is constructive: a finite automaton is associated with every formula and conversely [5]. We will thus reformulate Theorem 23 in terms of regular languages, i.e., sets of n -tuples of words recognized by finite automata. Again for details, see [3, 4].

Definition 24. Let $n \geq 1$. Let $U = (U_i)_{i \geq 0}$ be a Pisot numeration system. We first define the U -expansion of an n -tuple of integers as

$$\text{rep}_U(x_1, \dots, x_n) = (0^{\ell - |\text{rep}_U(x_1)|} \text{rep}_U(x_1), \dots, 0^{\ell - |\text{rep}_U(x_n)|} \text{rep}_U(x_n))$$

where $\ell = \max_{j \in \{1, \dots, n\}} |\text{rep}_U(x_j)|$.

Example 25. Consider the sequence $(F_i)_{i \geq 0}$ from Example 15. For the pair $(6, 10)$, we have $\text{rep}_F(6) = 1001$, $\text{rep}_F(10) = 10010$, and thus $\text{rep}_F(6, 10) = (01001, 10010)$.

The idea is to have n components of the same length; so the shortest expansions are padded to the left with zeroes. Hence an automaton can read simultaneously the i th digit of every component.

Definition 26. A set $X \subseteq \mathbb{N}^n$ is U -recognizable if there exists a deterministic finite automaton (DFA for short) reading n -tuples of digits and recognizing the set

$$\text{rep}_U(X) = \{\text{rep}_U(x_1, \dots, x_n) \mid (x_1, \dots, x_n) \in X\}.$$

If U is the usual base- k number system, we speak of k -recognizable sets.

We use standard conventions to represent DFA. The initial state has an incoming arrow. The final states have an outgoing arrow. Words are read from left to right, i.e., most significant digit first. If a transition is not depicted, then it leads to a dead state. Note that we allow leading zeroes (which does not affect recognizability by finite automaton).

Example 27. *The DFA depicted in Figure 1 recognizes the base-2 expansions of pairs (x, y) such that y is the largest power of 2 dividing x . For instance, $(1, 1)$, $(101010, 000010)$ or $(11100, 00100)$ belong to the recognized language. Otherwise stated, the set $\{(n, V_2(n)) \mid n > 0\}$ is 2-recognizable.*

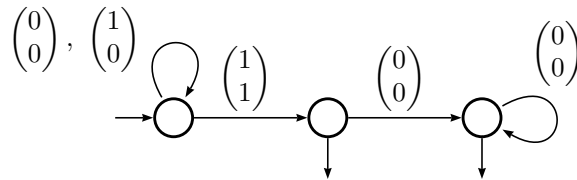


Figure 1: A DFA recognizing a language over $\{0, 1\}^2$.

Theorem 28. [3] *Let $n \geq 1$. Let $U = (U_i)_{i \geq 0}$ be a Pisot numeration system. A set $X \subseteq \mathbb{N}^n$ is U -recognizable if and only if it is U -definable.*

Example 29. *The DFA depicted in Figure 2 recognizes the T -expansions containing exactly two symbols 1. It means that the corresponding set of integers is T -recognizable. Recall that this set also is T -definable using the formula ψ of Example 20.*

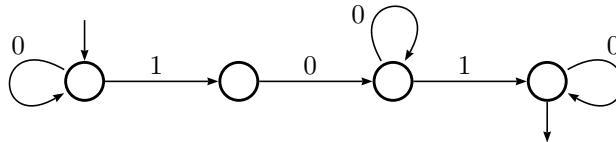


Figure 2: A DFA recognizing a T -definable set.

Thus our main theorem (Theorem 23) can also be restated as follows. We state this result for two sequences, but there is no problem to extend this result to n -tuples.

Theorem 30. *Let U be a Pisot numeration system. If a set P is U -recognizable, then it is decidable whether P is a 1-admissible set.*

The notion of k -synchronized sequences is classical [6]. The idea is that the graph of the function, i.e., the set of pairs $(n, f(n))_{n \geq 0}$, is k -recognizable. As we will easily see, synchronized sequences are sufficient to apply our theorem.

Definition 31. Let $U = (U_i)_{i \geq 0}$ be a Pisot numeration system. A sequence $(x_i)_{i \geq 0}$ of non-negative integers is U -synchronized if the set $\{(i, x_i) \mid i \geq 0\} \subset \mathbb{N}^2$ is U -recognizable.

Example 32. The DFA depicted in Figure 3 recognizes exactly the set $\{(i, 2i) \mid i \geq 0\}$. Otherwise stated, the sequence $(2i)_{i \geq 0}$ is 2-synchronized.

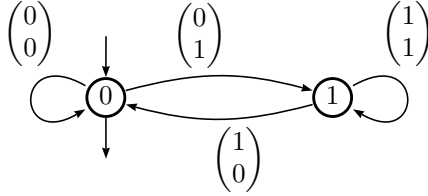


Figure 3: A DFA recognizing a 2-synchronized relation.

Corollary 33. Let $n \geq 1$. Let $U = (U_i)_{i \geq 0}$ be a Pisot numeration system. Let $(A_i)_{i \geq 0}$ and $(B_i)_{i \geq 0}$ be two U -synchronized sequences. Then the set $P = \{(A_k, B_k) \mid k \geq 0\}$ is U -recognizable, and thus it is decidable whether P is a 1-admissible set.

PROOF. The set of regular languages is closed under intersection and projection. The languages $\{\text{rep}_U(i, A_i) \mid i \geq 0\}$ and $\{\text{rep}_U(i, B_i) \mid i \geq 0\}$ are regular. Hence, one easily derives that the language $\{\text{rep}_U(A_i, B_i) \mid i \geq 0\}$ is also regular. Indeed, the languages $\{\text{rep}_U(A_i, i, j) \mid i, j \geq 0\}$ and $\{\text{rep}_U(j, i, B_i) \mid i, j \geq 0\}$ are regular. The intersection of these two sets is the regular language $\{\text{rep}_U(A_i, i, B_i) \mid i \geq 0\}$, and thus the projection on the first and third component also is.

6. Applications

Let $\mathcal{A} = \{1, \dots, n\}$ be a finite alphabet. An infinite word $w \in \mathcal{A}^{\mathbb{N}^{>0}}$ represents a subset of \mathbb{N}^n as follows. It is usual to assume that each symbol occurs infinitely often in w . We consider the n -tuple $(i_{m,1}, \dots, i_{m,n})$, where $i_{m,j}$ denotes the index of the m th symbol j occurring in w (as we did in the introduction for the Tribonacci game). Recall that the first letter in w has index 1.

Definition 34. We let P_w denote the subset of \mathbb{N}^n which is made up of all these n -tuples described above and also their permutations, and we add the n -tuple $\mathbf{0}$.

Therefore an infinite word w over $\mathcal{A} = \{1, \dots, n\}$ can be a convenient alternative to represent or characterize a set of \mathcal{P} -positions for a take-away game over n piles. Particularly when the infinite word is generated by a simple procedure such as iterating a morphism defined over a finite alphabet. For an example, see (1). Games such as Wythoff's game or the Tribonacci game have a set of \mathcal{P} -positions coded by a morphic word.

Example 35. *The infinite Fibonacci word $F = 1211212112112112112112 \dots$ is a fixed point of the morphism $1 \mapsto 12, 2 \mapsto 1$. It defines a subset P_F of \mathbb{N}^2 . The first few elements in P_F are*

$$(0, 0), (1, 2), (2, 1), (3, 5), (5, 3), (4, 7), (7, 4), (6, 10), (10, 6), (8, 13), (13, 8), (9, 15), (15, 9), \dots$$

It is well known that the n th \mathcal{P} -position of Wythoff's game is given by the position of the n th symbol 1 and n th symbol 2 occurring in F .

Remark 36. *Note that the use of infinite words also ensures that we get a partition of $\mathbb{N}_{>0}$. Assume that we have a binary alphabet $\{1, 2\}$. Let A_m (resp., B_m) be the position of the m th letter 1 (resp., 2) in an infinite word w . Then $\{A_m \mid m \geq 1\} \cup \{B_m \mid m \geq 1\} = \mathbb{N}_{>0}$ and $\{A_m \mid m \geq 1\} \cap \{B_m \mid m \geq 1\} = \emptyset$. In addition, we also consider the permutations of the n -tuples in Definition 34 since it provides "symmetric" rulesets, as it is the case in many take-away games.*

In what follows, we will give examples of sets P_w which satisfy the condition of Theorem 23, i.e., sets for which we can decide whether they are 1-admissible or not.

6.1. Periodic words

If we analyze the implementation of the decision procedure derived from Theorem 23, it is well known that the running time is bounded by an expression of the form

$$2^{2^{\dots 2^{p(n)}}}$$

where p is a polynomial. The height of tower of exponents corresponds to the number of nested quantifiers in the used logical formula (a quantifier leads to the construction of a non-deterministic automaton, and the subset algorithm for determinization explains this possible blow-up). One can doubt that such a bad complexity could lead to effective results. Yet, even though we have a very bad worst case scenario, positive results have been obtained in the field of combinatorics on words. See, for instance, [21] and Remark 22. It motivated us to obtain a different decision procedure (not based on the first order logic arithmetic) in some particular cases. Here we consider the special case of sets P_w defined by a binary periodic word $w = v^\omega = vvv \dots$. Note that our result should remain true for words over a larger alphabet.

We explain in details the case where v is a word on $\{1, 2\}$ that contains as many ones than twos. Let $\ell = |v|$ denote the length of v . We let $|v|_a$ denote the number of occurrences of the symbol a within v . Hence we assume that $|v|_1 = |v|_2 = \ell/2$. The basic idea is that the periodicity allows us to check the 1-admissibility only on the small square $\llbracket 0, \ell \rrbracket^2$. The set P_w is periodic in the following sense. Let $\mathbf{p} \in P_w$. If $\mathbf{p} \neq (0, 0)$, then $\mathbf{p} + (\ell, \ell) \in P_w$. Indeed, if $\mathbf{p} = (A_m, B_m)$ (with the notations of Remark 36), then $\mathbf{p} + (\ell, \ell) = (A_{m+\ell/2}, B_{m+\ell/2})$. Similarly, if $\mathbf{p} > (\ell, \ell)$, then $\mathbf{p} - (\ell, \ell) \in P_w$. In other words, if P_w^0 denotes the set $P_w \cap \llbracket 1, \ell \rrbracket^2$, then

$$P_w = \{(0, 0)\} \cup (P_w^0 + \{k \cdot (\ell, \ell) \mid k \in \mathbb{N}\}).$$

Furthermore, let s be the unique integer such that $s\frac{\ell}{2} < m \leq (s+1)\frac{\ell}{2}$. Then $\mathbf{p} = (A_m, B_m) \in \llbracket s\ell+1, (s+1)\ell \rrbracket^2$. Indeed, the m th occurrence of 1 necessarily appears in w in the occurrence $s+1$ of u . The set $P_w - P_w$ has also some periodicity as expressed in the next lemma.

Lemma 37. *Let w be a binary periodic word on $\{1, 2\}$. Let v be the period of w and let ℓ be the length of v . Assume that v has the same number of ones and twos. We have*

$$(P_w - P_w) \cap \mathbb{N}^2 = (P_w \cap \llbracket 0, 2\ell \rrbracket^2 - P_w \cap \llbracket 0, \ell \rrbracket^2) \cap \mathbb{N}^2 + \{k \cdot (\ell, \ell) \mid k \in \mathbb{N}\}.$$

PROOF. Let $\mathbf{m} \in (P_w - P_w) \cap \mathbb{N}^2$. We can write $\mathbf{m} = \mathbf{p}_1 - \mathbf{p}_2$ with $\mathbf{p}_1 \geq \mathbf{p}_2$ both in P_w . As noticed before, \mathbf{p}_1 (respectively \mathbf{p}_2) lies in a set $\llbracket s\ell+1, (s+1)\ell \rrbracket^2$ for some s (respectively in $\llbracket t\ell+1, (t+1)\ell \rrbracket^2$).

Since $\mathbf{p}_1 \geq \mathbf{p}_2$, we have $s \geq t$. We can write, if $s \neq t$,

$$\mathbf{m} = (\mathbf{p}_1 - (s-1) \cdot (\ell, \ell)) - (\mathbf{p}_2 - t \cdot (\ell, \ell)) + (s-t-1) \cdot (\ell, \ell)$$

and if $s = t$,

$$\mathbf{m} = (\mathbf{p}_1 - s \cdot (\ell, \ell)) - (\mathbf{p}_2 - s \cdot (\ell, \ell)).$$

Since $\mathbf{p}_1 - s \cdot (\ell, \ell)$ and all the similar positions in the previous expressions are elements of P_w , we are done.

Assume now that $\mathbf{m} \in (P_w \cap \llbracket 0, 2\ell \rrbracket^2 - P_w \cap \llbracket 0, \ell \rrbracket^2) + \{k \cdot (\ell, \ell) \mid k \in \mathbb{N}\}$. Then $\mathbf{m} = \mathbf{p}_1 - \mathbf{p}_2 + k \cdot (\ell, \ell)$ for some $\mathbf{p}_1, \mathbf{p}_2$ and k . If $\mathbf{m} \neq (0, 0)$, then $\mathbf{p}_1 \neq (0, 0)$ and $\mathbf{p}_1 + k \cdot (\ell, \ell) \in P_w$. Thus $\mathbf{m} \in P_w - P_w$.

The next lemma says that we only need to check the admissibility on the small positions.

Lemma 38. *Let w be a binary periodic word on $\{1, 2\}$. Let v be the period of w and let ℓ be the length of v . Assume that v has the same number of ones and twos. Then the set P_w is 1-admissible if and only if for each position \mathbf{x} in $\llbracket 0, 2\ell \rrbracket^2$ not in P_w , there is a position $\mathbf{p} \in P_w$ such that $\mathbf{p} \leq \mathbf{x}$ and $\mathbf{x} - \mathbf{p} \notin P_w - P_w$.*

PROOF. By Lemma 12, P_w is 1-admissible if and only if the game $H(P_w)$ is 1-invariant. This is equivalent to say that for any position $\mathbf{x} \notin P_w$, there is a position $\mathbf{p} \in P_w$ such that $\mathbf{p} \leq \mathbf{x}$ and $\mathbf{x} - \mathbf{p} \notin P_w - P_w$. Therefore assume that the latter fact is true for $\mathbf{x} \in \llbracket 0, 2\ell \rrbracket^2 \setminus P_w$. To prove the lemma we need to prove the fact for any $\mathbf{x} \notin P_w$. We thus can assume that $\mathbf{x} \notin \llbracket 0, 2\ell \rrbracket^2$.

Assume first that there exists an element \mathbf{p} of P_w such that $\mathbf{m} = \mathbf{x} - \mathbf{p}$ is equal to $(0, k)$ or $(k, 0)$ for some k . Since the two sets $\{A_m, m \in \mathbb{N}_{>0}\}$ and $\{B_m, m \in \mathbb{N}_{>0}\}$ form a partition of $\mathbb{N}_{>0}$, we have $\mathbf{m} \notin P_w - P_w$, and we are done.

Assume now that this is not the case. Since the two sets $\{A_m, m \in \mathbb{N}_{>0}\}$ and $\{B_m, m \in \mathbb{N}_{>0}\}$ form a partition, we must have, without loss of generality, $\mathbf{x} = (A_m, y)$ for some m, y . Since $(A_m, B_m) - \mathbf{x} \neq (0, k)$ for $k \geq 0$, we have $y < B_m$. In the same way, we also have $\mathbf{x} = (x, A_{m'})$ for some m', x , and

$x < B_{m'}$ (it is not possible to have $y = B_{m'}$ since the positions (A_k, B_k) are strictly increasing). Finally $\mathbf{x} = (A_m, A_{m'})$ with $A_m < B_{m'}$ and $A_{m'} < B_m$. The two positions (A_m, B_m) and $(B_{m'}, A_{m'})$ are necessarily in the same square $\llbracket sl+1, (s+1)\ell \rrbracket^2$ for some s . Hence \mathbf{x} is also in this square and since $\mathbf{x} \notin \llbracket 0, 2\ell \rrbracket^2$, we have $s \geq 2$.

We now consider the position $\mathbf{x}' = \mathbf{x} - (s-1) \cdot (\ell, \ell)$. We have $\mathbf{x}' \in \llbracket 0, 2\ell \rrbracket^2 \setminus P_w$. By hypothesis, there exists a position $\mathbf{p}' \in P_w$ such that $\mathbf{m}' = \mathbf{x}' - \mathbf{p}' \notin P_w - P_w$. If $\mathbf{p}' \neq (0, 0)$, we play the move $\mathbf{m} = \mathbf{m}'$ from \mathbf{x} to the position $\mathbf{p} = \mathbf{p}' + (s-1) \cdot (\ell, \ell)$. Clearly $\mathbf{p} \in P_w$.

If $\mathbf{p}' = (0, 0)$, then $\mathbf{x}' \notin P_w - P_w$, and we will play the move $\mathbf{m} = \mathbf{x}$ to $\mathbf{p} = (0, 0)$. Assume for the contradiction that $\mathbf{m} \in P_w - P_w$. Then by Lemma 37, $\mathbf{x} = \mathbf{p}_1 - \mathbf{p}_2 + t \cdot (\ell, \ell)$ for some $\mathbf{p}_1, \mathbf{p}_2$ and t . But we also have $\mathbf{x} = \mathbf{x}' + (s-1) \cdot (\ell, \ell)$. Since $\mathbf{p}_1 - \mathbf{p}_2 \in \llbracket 0, 2\ell \rrbracket^2$ and $\mathbf{x}' \in \llbracket \ell+1, 2\ell \rrbracket^2$, we have $s = t$ or $s = t + 1$. Then necessarily, $\mathbf{x}' = \mathbf{p}_1 - \mathbf{p}_2 + \delta \cdot (\ell, \ell)$ with $\delta \in \{0, 1\}$. This is a contradiction since $\mathbf{x}' \notin P_w - P_w$.

A natural algorithm can be deduced from the previous lemma to decide the 1-admissibility of P_w . Indeed, it is enough to consider the (at most) $4\ell^2$ elements \mathbf{x} of $\llbracket 0, 2\ell \rrbracket^2 \setminus P_w$. For each of them, consider the at most $2\ell + 1$ possible positions $\mathbf{p} \leq \mathbf{x}$ in P_w and check if $\mathbf{m} = \mathbf{x} - \mathbf{p} \in P_w - P_w$. By Lemma 37, since $\mathbf{m} \in \llbracket 0, 2\ell \rrbracket^2$, if $\mathbf{m} \in P_w - P_w$, then

$$\mathbf{m} \in (P_w \cap \llbracket 0, \ell \rrbracket^2 - P_w \cap \llbracket 0, \ell \rrbracket^2) \cap \mathbb{N}^2 + \{(0, 0), (\ell, \ell)\}.$$

We can compute this latter set once for all the procedure in $O(\ell^2)$ steps. Thus the total procedure runs in polynomial time in ℓ .

Corollary 39. *Let w be a binary periodic word on $\{1, 2\}$ with period v . Assume that v has the same number of ones and twos. There is an algorithm running in polynomial time in $|v|$ that decides if the set P_w is 1-admissible.*

Remark 40. *If v does not contain the same number of ones and twos, we still have some periodicity in P_w and $P_w - P_w$. Assume that $|v|_1 < |v|_2$. Let r be the least common multiple of $|v|_1$ and $|v|_2$. Let $\mathbf{u} = (\frac{r}{|v|_1} \cdot |v|, \frac{r}{|v|_2} \cdot |v|)$. Then one can check that for any $m > 0$, $(A_m, B_m) + \mathbf{u} = (A_{m+r}, B_{m+r})$ and thus is an element of P_w . Let $\mathbf{u}^{\mathbf{R}}$ be the symmetric of \mathbf{u} obtained by permuting its coordinates. We have:*

$$P_w = \{(0, 0)\} \cup (\{(A_m, B_m) \mid 1 \leq m \leq r\} + \mathbb{N}\mathbf{u}) \cup (\{(B_m, A_m) \mid 1 \leq m \leq r\} + \mathbb{N}\mathbf{u}^{\mathbf{R}}).$$

The set $P_w - P_w$ can be seen as the union of four different parts D_1, D_2, D_3 and D_4 . The first part D_1 is similar to the previous case and corresponds to the difference between two elements of P_w on the same form (A_m, B_m) (with the convention that $(A_0, B_0) = (0, 0)$):

$$D_1 = \{(A_m - A_p, B_m - B_p) \mid 0 \leq p \leq m \leq 2r \text{ and } p \leq r\} + \mathbb{N}\mathbf{u}.$$

The set D_2 is the symmetric of D_1 , it corresponds to the differences between two elements (B_m, A_m) . The set D_3 corresponds to the difference between a position (A_m, B_m) and a position (B_p, A_p) :

$$D_3 = (\{(A_m - B_p, B_m - A_p) \mid 0 \leq p \leq m \leq (k_0 + 1)r \text{ and } p \leq r\} + \{k_1 \mathbf{u} - k_2 \mathbf{u}^{\mathbf{R}} \mid k_1, k_2 \in \mathbb{N}\}) \cap \mathbb{N}^2$$

where k_0 is such that $k_0 \geq \frac{|v|_2}{|v|_1}$. Finally, the set D_4 is the symmetric of D_3 .

With the same argument as before, it should be enough to check the 1-admissibility on the set $\llbracket 0, (k_0 + 1)r \rrbracket^2$. This would lead to a polynomial-time algorithm (in the length of $|v|$).

Remark 41. If $w = v^\omega$, then P_w is definable in $\langle \mathbb{N}, + \rangle$. So instead of our specific algorithm, one can also use the general procedure given by Theorem 23.

6.2. Parikh-constant morphic words

We turn to a slightly more general situation than $w = v^\omega$ and consider words of a special form obtained by iterating a morphism. Here, the infinite word w is of the form $v_1 v_2 v_3 \dots$ and every finite word v_i is a permutation of the letters of every v_j . Let u be a finite word. A morphism $f : \mathcal{A}^* \rightarrow \mathcal{A}^*$ is ℓ -uniform if, for all $a \in \mathcal{A}$, $|f(a)| = \ell$. A morphism $f : \mathcal{A}^* \rightarrow \mathcal{A}^*$ is *Parikh-constant* if, for all $a, b, c \in \mathcal{A}$, $|f(a)|_c = |f(b)|_c$. Note that if f is Parikh-constant, for all $a, b \in \mathcal{A}$, $|f(a)| = |f(b)|$, i.e., f is ℓ -uniform for some ℓ .

Lemma 42. [4] Let $\ell \geq 2$ be an integer. Let $w = w_1 w_2 w_3 \dots \in \mathcal{A}^{\mathbb{N}^{>0}}$ be a fixed point of a ℓ -uniform morphism. For every $a \in \mathcal{A}$, the set $\{i > 0 \mid w_i = a\}$ is ℓ -definable.

We let $\chi_{w,a}$ denote the formula defining the set $\{i > 0 \mid w_i = a\}$ given in the above lemma. Otherwise stated, $\chi_{w,a}(n)$ holds if and only if n belongs to that set.

Proposition 43. Let $w = w_1 w_2 w_3 \dots \in \mathcal{A}^{\mathbb{N}^{>0}}$ be a fixed point of a Parikh-constant morphism $f : \mathcal{A}^* \rightarrow \mathcal{A}^*$. One can decide whether P_w is a 1-admissible subset of $\mathbb{N}^{\#\mathcal{A}}$.

PROOF. The morphism f is ℓ -uniform for an ℓ . From Theorem 23, we simply need to prove that P_w is ℓ -definable. Since f is Parikh-constant, note that $\ell = |f(a)|$ and $N_b := |f(a)|_b$, $b \in \mathcal{A}$, are given constants depending only on f and not on a . We may assume that $N_b \geq 1$ for all b , otherwise we can restrict the morphism to a smaller alphabet. Let $a, b \in \mathcal{A}$ and $r \in \llbracket 1, N_a \rrbracket$. We let $C_{r,a,b}$ be the position of the r th symbol a occurring in $f(b)$. Note that $C_{r,a,b}$ is a constant in $\llbracket 1, \ell \rrbracket$ which is also derived from f .

We define a predicate $g_{w,a}(m, j)$ which holds if and only if the position of the m th symbol a occurring in w is j ,

$$g_{w,a}(m, j) \equiv (\exists q)(\exists r)(m = q \cdot N_a + r \wedge 0 < r \leq N_a \wedge \bigvee_{b \in \mathcal{A}} (\chi_{w,b}(q+1) \wedge j = q \cdot \ell + C_{r,a,b})).$$

Indeed, w is the (infinite) concatenation of blocks of length ℓ and each such block contains exactly N_a letters a . If $m = qN_a + r$ with $0 < r \leq N_a$, then the m th occurrence of a appears in the $(q+1)$ st block of length ℓ . Since $f(w) = w$, this block is equal to $f(w_{q+1})$. The disjunction expresses the fact that $w_{q+1} = b$ for some b , and thus the m th letter a occurs in position $q\ell + C_{r,a,b}$.

Example 44. Consider the morphism $f : 1 \mapsto 112, 2 \mapsto 121$. A prefix of the fixed point w of f is

112112121112112121112121112...

The first few elements in P_w are $(0, 0), (1, 3), (2, 6), (4, 8), (5, 12), (7, 15), (9, 17), \dots$. With the notation of the previous proof, $g_{w,1}(6, 9)$ and $g_{w,2}(6, 17)$ hold. We have $\ell = 3$, $N_1 = 2$ and $N_2 = 1$. Thus $6 = 2N_1 + 2$. This means that the sixth 1 occurs in the third block of length 3 which is $f(w_3) = f(2) = 121$, and $C_{1,1,2} = 1$ and $C_{2,1,2} = 3$. Hence, the position of the sixth 1 is $2 \cdot L + C_{2,1,2} = 9$.

6.3. Recurrence of order 2

In [14] Fraenkel considered a class of games whose \mathcal{P} -positions can be characterized using a numeration system $(U_i)_{i \geq 0}$ satisfying, for $i \geq 1$, the relation

$$U_i = (s + t - 1)U_{i-1} + sU_{i-2} \quad (2)$$

where $U_{-1} = 1/s$, $U_0 = 1$ and $s, t \in \mathbb{N}$ are positive. The following proposition is easy to prove. It implies that the corresponding numeration system is a Pisot system.

Proposition 45. Let $s, t > 0$ be integers. The positive root of $X^2 - (s + t - 1)X - s$ is a Pisot number. In particular, the numeration system defined by (2) is a Pisot numeration system.

The take-away game devised by Fraenkel in [14, Theorem 5.1] has the following property: a pair (x, y) is a \mathcal{P} -position if and only if

- $\text{rep}_U(x)$ ends in an even (possibly 0) number of zeroes,
- $\text{rep}_U(y) = \text{rep}_U(x)0$.

This syntactical property permits us to state the following result.

Proposition 46. For the take-away game in [14], the set of \mathcal{P} -positions is U -recognizable and thus, one can decide whether this set is 1-admissible.

PROOF. Since we have a Pisot numeration system, the set \mathbb{N} is U -recognizable [3]. Otherwise stated, there exists a DFA accepting $\text{rep}_U(\mathbb{N})$. One can obviously modify this automaton to accept exactly the U -expansions ending with an even number of zeroes.

We can also construct a DFA recognizing pairs of words of the form $(0u, u0)$ where u is a word over the alphabet A . The set of states is A . The initial

state is 0. This state is also final. States are used to store the last letter that was read on the second component. Reading $(a, b) \in A^2$ from state a leads to state b . Reading (a, b) from state $c \neq a$ leads to a dead state. Such a DFA for $A = \{0, 1\}$ is depicted in Figure 3.

From these two automata, one can derive a DFA recognizing pairs of words of the form $(0u, u0)$ where u is a U -expansion ending with an even number of zeroes. To finish the proof, one also has to consider the symmetric version for words of the form $(u0, 0u)$.

Remark 47. For $s = t = 1$, we are back to the Fibonacci sequence and Wythoff's game. The DFA accepting pairs of the forms $(0u, u0)$ where u is a F -expansion ending with an even number of zeroes is depicted in Figure 4.

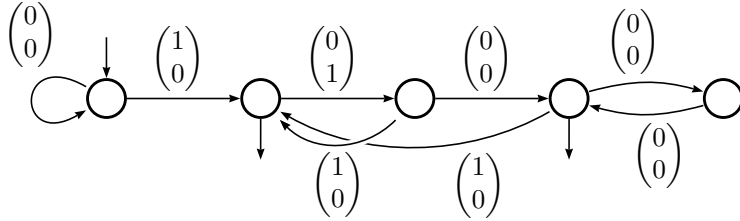


Figure 4: The set of \mathcal{P} -positions of Wythoff's game is F -recognizable.

6.4. Sets derived from \mathcal{P} -positions of existing games

Here we will observe that many games like Wythoff's game, Tribonacci and Pisot unit game, Raleigh game, etc. have a U -recognizable set of \mathcal{P} -positions, and thus Theorem 23 can be applied.

- For Wythoff's game, with the strong syntactical properties of the F -representations of the \mathcal{P} -positions, we may apply Proposition 46 and get the DFA depicted in Figure 4.
- A similar argument may be developed for the Tribonacci game because of the next statement. Note that this game has a set of \mathcal{P} -positions coded by P_T where T is the word given in (1).

Theorem 48. [10, Theorem 5.2] The triple (a, b, c) with $a < b < c$ is a \mathcal{P} -position of the Tribonacci game if and only if $\text{rep}_T(a - 1) = w0$, $\text{rep}_T(b - 1) = w01$ and $\text{rep}_T(c - 1) = w011$, where rep_T is the T -expansion associated with the Tribonacci sequence given in Example 16.

Example 49. Table 2 illustrates the previous result. A_m (respectively B_m and C_m) denotes the position of the m th 1 (resp. 2 and 3) in the Tribonacci word.

Corollary 50. It is decidable whether the set of \mathcal{P} -positions of the Tribonacci game [10] is 1-admissible.

A_m	B_m	C_m	$\text{rep}_T(A_m - 1)$	$\text{rep}_T(B_m - 1)$	$\text{rep}_T(C_m - 1)$
1	2	4	ε	1	11
3	6	11	10	101	1011
5	9	17	100	1001	10011
7	13	24	110	1101	11011

Table 2: First \mathcal{P} -positions and the corresponding Tribonacci-expansions.

PROOF. One can adapt the proof of Proposition 46 showing that the set of \mathcal{P} -positions is T -recognizable. One has to intersect the DFA in Figure 5 with a DFA recognizing valid T -expansions for the first component. Note that $\text{rep}_U(\mathbb{N})$ is the set of words over $\{0, 1\}$ not containing the factor 111. Also, for every Pisot numeration system U , the set $\{(i, i+1) \mid i \geq 0\}$ is U -recognizable [19, 20]. Thus the difference of 1 that appears in Theorem 48 can easily be handled by finite automata. An alternative explanation is to make the following observation. Let X be a U -recognizable subset of \mathbb{N}^3 . By Theorem 28, it is equivalent to the fact that X is U -definable by a formula φ . Now, the set X' defined by $(a, b, c) \in X \Leftrightarrow (a+1, b+1, c+1) \in X'$ is trivially U -definable by a formula φ' because the successor is definable in $\langle \mathbb{N}, + \rangle$; see Remark 19. Indeed $\varphi'(a', b', c')$ holds if and only if there exist a, b, c such that $\varphi(a, b, c)$ and $a' = a+1, b' = b+1, c' = c+1$.

Example 51. In the DFA depicted in Figure 5, reading (a, b, c) from state ab leads to state bc . States are used to store the last symbol read on the second and third component. We have duplicated the state 11 to take into account that on the third component, we must accept words ending with 011 and not those ending with 111. Note that this DFA does not test the occurrence of a factor 111.

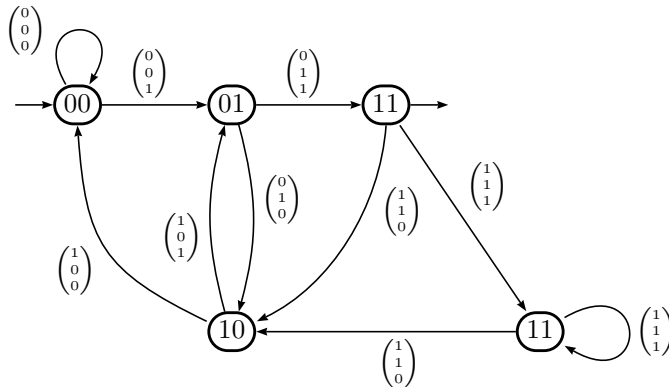


Figure 5: A DFA recognizing $(00w0, 0w01, w011)$.

- In [11] the considered games have a set of \mathcal{P} -positions coded by generalized

Tribonacci words that are a fixed point of the morphism $1 \mapsto 1^s 2, 2 \mapsto 13, 3 \mapsto 1$ where $s \geq 1$. We let G_s denote this game (for a fixed value of the parameter s). In this setting, one considers the sequence $(U_i)_{i \geq 0}$ where

$$U_{i+3} = sU_{i+2} + U_{i+1} + U_i \quad (3)$$

with the initial conditions $U_0 = 1, U_1 = sU_0 + 1, U_2 = sU_1 + U_0 + 1$. The language of all U -expansions is recognized by the DFA depicted in Figure 6

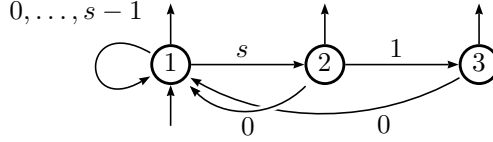


Figure 6: A DFA accepting all the U -representations.

The following theorem is a natural generalization of Theorem 48.

Theorem 52. [11, Section 4] *Let $s \geq 1$. The triple (a, b, c) with $a < b < c$ is a \mathcal{P} -position of the game G_s if and only if $\text{rep}_U(a-1) = w, \text{rep}_U(b-1) = ws, \text{rep}_U(c-1) = ws1$ and w is the label of a cycle starting from the initial state 1 in the DFA depicted in Figure 6, where rep_U is the U -expansion associated with the sequence (3).*

Example 53. *Table 3 illustrates the previous result. For $s = 3$, the set of \mathcal{P} -positions of G_3 is coded by the word*

11121112111213111211121112131112111213111213111211112...

The numeration systems is constructed over the sequence 1, 4, 14, 47, 159, ...

A_m	B_m	C_m	$\text{rep}_T(A_m - 1)$	$\text{rep}_T(B_m - 1)$	$\text{rep}_T(C_m - 1)$
1	4	14	ε	3	31
2	8	28	1	13	131
3	12	42	2	23	231
5	18	61	10	103	1031

Table 3: First \mathcal{P} -positions of G_3 and the corresponding expansions.

Corollary 54. *Let $s \geq 1$. It is decidable whether the set of \mathcal{P} -positions of the game G_s [11] is 1-admissible.*

PROOF. Again the proof is similar to the one of Proposition 46 and Corollary 50. One has to devise a DFA for words of the form $(00u, 0u3, u31)$. Moreover, testing if u labels a cycle can also be handled by a DFA. Finally, the positive root of $X^3 - sX^2 - X - 1$ is again a Pisot number, so the set $\{(i, i+1) \mid i \geq 0\}$ is U -recognizable [19, 20].

- The Raleigh game [17] is a variant game played on three piles of tokens. Again this game has a set of \mathcal{P} -positions that is F -recognizable. First we provide a new (morphic) characterization of its set of \mathcal{P} -positions.

Lemma 55. *The set of \mathcal{P} -positions of the Raleigh game is coded by the fixed point $12312123 \cdots$ of the morphism $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 12$.*

PROOF. Let w be the fixed point of the morphism $f : 1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 12$. In [17], the \mathcal{P} -positions of the Raleigh game (A_m, B_m and C_m are defined as before) are characterized according to the following relations: $A_1 = 1, B_1 = 2, C_1 = 3$ and for all $m > 1$,

$$\begin{aligned} A_m &= \text{mex}\{A_i, B_i, C_i : 0 \leq i < m\} \\ B_m &= A_m + 1 \\ C_m &= \begin{cases} C_{m-1} + 3, & \text{if } A_m - A_{m-1} = 2; \\ C_{m-1} + 5, & \text{otherwise.} \end{cases} \end{aligned}$$

It is not hard so see that w is the unique word over $\{1, 2, 3\}$ which exactly corresponds to the sequence (A_m, B_m, C_m) . Indeed, the definition of f implies that for all $m \geq 1$, the m th 1 of w appears before the m th 2 and the m th 3. In other words, the 1's in w correspond to the values A_m . By definition of f , one can also observe that the m th 1 and the m th 2 are always successors in w . Thus the 2's of w correspond to B_m . Concerning the 3's of w , there are only "produced" as images of values 2. In other words, the m th 2 produces the m th 3. Since the gap between two 2's is either 2 or 3, and as $|f(1)| = |f(3)| = 2$, the difference between the m th and $(m+1)$ st 3 is equal to 3 or 5, according to the gap between the m th and $(m+1)$ st 2 (which is identical to the gap between the m th and $(m+1)$ st 1).

The DFA associated with the morphism $f : 1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 12$ is depicted in Figure 7. It is defined as follows (and this definition can be extended to any morphism). Its set of states is $\{1, 2, 3\}$. If $f(a) = bc$, then the DFA has an edge from a to b (resp., c) of label 0 (resp., 1). If $f(a) = b$, then the DFA has an edge from a to b of label 0. The initial state is 1 because the word is obtained by iterating f from 1. Note that this DFA has a synchronizing property. Reading 00 (resp., 01, 010) from every state leads to state 1 (resp., 2, 3).

We recall the following general result. See, for instance, [2, Section 3.4]

Proposition 56. [28] *Let A be a finite alphabet. Let $f : A^* \rightarrow A^*$ be a morphism generating an infinite word w when iterating f on a symbol $a \in A$. Let M be the language of words not starting with 0 and accepted by automaton associated with f where a is the initial state and all states are final. The m th symbol of w (where indexing start with 0) is the state reached from a when reading the $(m+1)$ st word of M genealogically ordered.*

Example 57. *Observe that the DFA depicted in Figure 7 recognizes exactly the F -expansions of the integers. The first few words (not starting by zero) accepted*

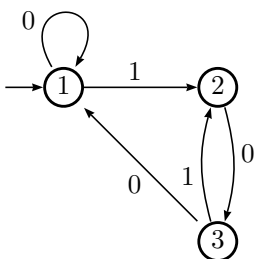


Figure 7: The DFA associated with the Raleigh morphism.

by this automaton (the first elements in M) are given below. When reading these words from state 1, we have indicated the corresponding reached state.

ε	1	10	100	101	1000	1001	1010
	1	2	3	1	2	1	2
							3

The reader may observe that we have a prefix of the infinite word generated by f . So to get the m th symbol (where the first symbol has index 0) of this word, one has to feed the DFA with $\text{rep}_F(m)$.

Theorem 58. *The 3-tuple (a, b, c) with $a < b < c$ is a \mathcal{P} -position of the Raleigh game if and only if $\text{rep}_F(a - 1) = w00$, $\text{rep}_F(b - 1) = w01$, $\text{rep}_F(c - 1) = w010$ where rep_F is the F -expansion associated with the Fibonacci sequence.*

Example 59. *Table 4 illustrates the previous result. It is pretty easy to enumerate the \mathcal{P} -positions. Enumerate (by genealogical ordering) the words over $\{0, 1\}$ avoiding the factor 11 and ending with 0: 10, 100, 1000, 1010, ... For the m th element w in this enumeration, consider the 3-tuple of words $(w0, w1, w10)$. Note that the three added suffixes all start with 0.*

A_m	B_m	C_m	$\text{rep}_F(A_m - 1)$	$\text{rep}_F(B_m - 1)$	$\text{rep}_F(C_m - 1)$
1	2	3	ε	1	10
4	5	8	100	101	1010
6	7	11	1000	1001	10010
9	10	16	10000	10001	100010
12	13	21	10100	10101	101010

Table 4: First \mathcal{P} -positions of the Raleigh game and the corresponding F -expansions.

PROOF. We make use of Proposition 56. Let M be the language made of the words not starting with 0 and accepted by the automaton depicted in Figure 7 where all states are final. Note that $M = \text{rep}_F(\mathbb{N})$. Hence the state reached when reading $\text{rep}_F(m)$ provides the m th symbol of the word (index starting with 0).

The set $M \cap \{0, 1\}^{\geq 4}$ is partitioned into four sets F_1, F_2, F_3 of words of length at least four: those ending 00, 01 and 010 respectively. Thus for all $m \geq 5$, $\text{rep}_F(m)$ belongs to one of these sets. Moreover, if $\text{rep}_F(m)$ belongs to F_i , then the m th symbol of the word is i .

Since M is prefix-closed, we know that if w and w' are two consecutive words in M of length at least two (for the genealogical ordering), then $w00$ and $w'00$ are consecutive elements in F_1 . From Proposition 56, they correspond to two consecutive occurrences of the symbol 1 in the infinite word. The same observation is made for elements in F_2, F_3 (for the words $w01, w'01$ and $w010, w'010$ respectively).

From this observation, to finish the proof by induction — enumerating by increasing genealogical order the words of M — we just need to find one 3-tuple (a, b, c) which is a \mathcal{P} -position of the Raleigh game and such that $\text{rep}_F(a-1, b-1, c-1) = (u00, u01, u010)$. One can take $(6, 7, 11)$. We have

$$\text{rep}_F(5, 6, 10) = (u00, u01, u010) \text{ with } u = 10.$$

The first few values have thus to be checked by direct inspection.

Corollary 60. *It is decidable whether the set of \mathcal{P} -positions of the Raleigh game is 1-admissible.*

Remark 61. *One can redo the proof of Theorem 52 in a way similar to the proof of Theorem 58. Let M be the language made of the words not starting with 0 and accepted by the automaton depicted in Figure 6. The set $M \cap \{0, 1\}^{\geq 2}$ is partitioned into three sets F_1, F_2, F_3 of words of length at least two: those not ending with s nor $s1$, those ending with s , and those ending with $s1$ respectively.*

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