A coupled Electro-Thermo-Mechanical Discontinuous Galerkin method applied on composite materials

Lina Homsi, Ludovic Noels
LTAS/CM3, University of Liège
Belgium

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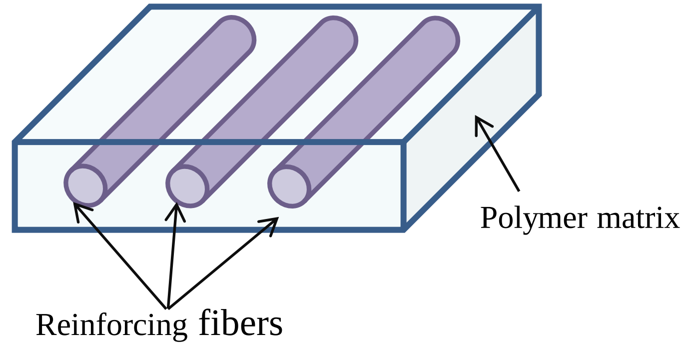
Université de Liège

Introduction

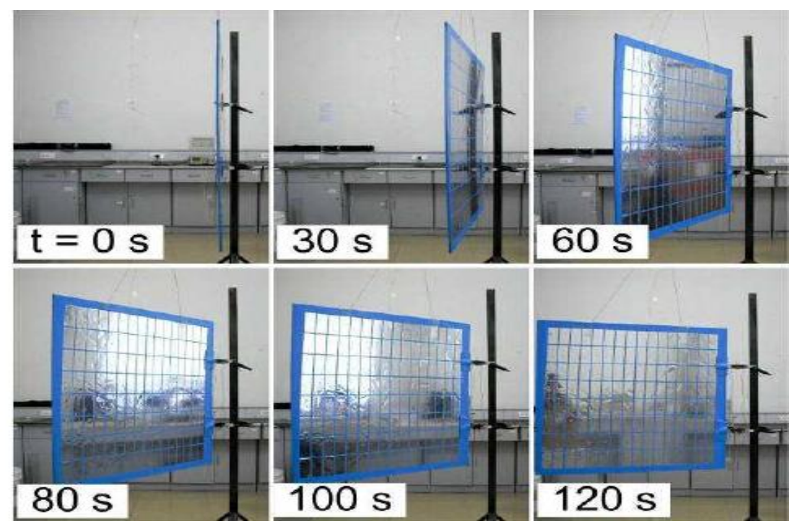
Carbon fiber polymer composites

- Multifuctional materials
  - Structural capabilities
  - Electrical and thermal functions



Carbon fiber reinforced composite

Application: Activation of fiber reinforced shape memory polymer composites in its application in deployable hinge in space [1].



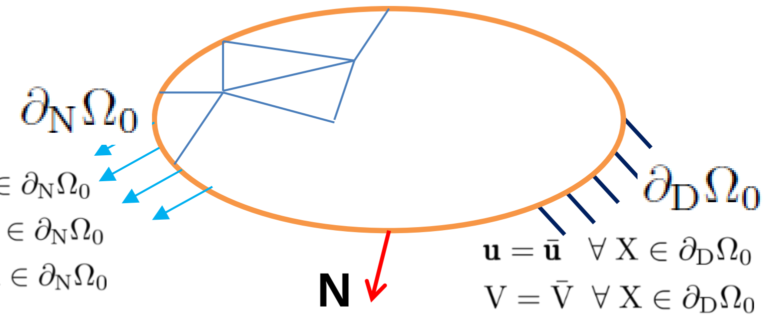
Shape recovery process of a prototype of solar array actuated by SMPC hinge

[1] L. Xin and Z. Ruirui. Smart materials and structures, 2011

Outline

- Introduction
  - Constitutive equations
  - Main concept and equation of Discontinuous Galerkin (DG) method
- DG Formulation for Electro-Thermo-Mechanical coupled problem
  - Weak form of equations
  - Numerical properties i.e. solution uniqueness, convergence rate…
- Numerical examples
- Conclusions & Perspectives

Governing equations for Electro-Thermo-Mechanical coupling



$$\mathbf{P} \cdot \mathbf{N} = \bar{\mathbf{T}} \quad \forall\, \mathrm{X} \in \partial_N \Omega_0$$
$$\mathbf{J}_e \cdot \mathbf{N} = \bar{\mathrm{J}}_e \quad \forall\, \mathrm{X} \in \partial_N \Omega_0$$
$$\mathbf{J}_y \cdot \mathbf{N} = \bar{\mathrm{J}}_y \quad \forall\, \mathrm{X} \in \partial_N \Omega_0$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \forall\, \mathrm{X} \in \partial_D \Omega_0$$
$$\mathrm{V} = \bar{\mathrm{V}} \quad \forall\, \mathrm{X} \in \partial_D \Omega_0$$
$$\mathrm{T} = \bar{\mathrm{T}} \quad \forall\, \mathrm{X} \in \partial_D \Omega_0$$

$$\forall\, \mathbf{u},\, \mathrm{V},\, \mathrm{T} \in \left[\mathrm{H}^2(\Omega_0)\right]^3 \times \mathrm{H}^2(\Omega_0) \times \mathrm{H}^{2^+}(\Omega_0)$$

Conservation of momentum balance	Conservation of electric charge	Conservation of energy
$\nabla_0 \cdot \mathbf{P}^T = 0 \quad \forall\, \mathrm{X} \in \Omega_0$	$\nabla_0 \cdot \mathbf{J}_e = 0 \quad \forall\, \mathrm{X} \in \Omega_0$	$\nabla_0 \cdot \mathbf{J}_y = 0 \quad \forall\, \mathrm{X} \in \Omega_0$
$\mathbf{P} = \mathbb{P}(\mathbf{F}, \mathrm{T}, \mathbf{I})$	$\mathbf{J}_e = \mathbb{J}_e(\mathbf{F}, \mathrm{V}, \mathrm{T})$	$\mathbf{J}_y = \mathbb{J}_y(\mathbf{F}, \mathrm{V}, \mathrm{T})$

Electric current density $\quad \mathbf{J}_e = \mathbf{L} \cdot (-\nabla_0 \mathrm{V}) + \alpha \mathbf{L} \cdot (-\nabla_0 \mathrm{T})$

Energy flux $\quad \mathbf{J}_y = \mathbf{Q} + \mathrm{V}\, \mathbf{J}_e$

Heat flux $\quad \mathbf{Q} = \mathbf{K} \cdot (-\nabla_0 \mathrm{T}) + \alpha \mathrm{T}\, \mathbf{J}_e$

Electro-Thermal constitutive relations

- Vector of the unknown fields: $\begin{pmatrix} \mathrm{f_V} \\ \mathrm{f_T} \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{V}}{\mathrm{T}} \\ \frac{1}{\mathrm{T}} \end{pmatrix}$
- Matrix form of fluxes and fields gradient

$$\begin{pmatrix} \mathbf{J}_\mathrm{e} \\ \mathbf{J}_\mathrm{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathrm{f_T}}\mathbf{L} & -\frac{\mathrm{f_V}}{\mathrm{f_T^2}}\mathbf{L} + \alpha\frac{1}{\mathrm{f_T^2}}\mathbf{L} \\ -\frac{\mathrm{f_V}}{\mathrm{f_T^2}}\mathbf{L} + \alpha\frac{1}{\mathrm{f_T^2}}\mathbf{L} & \frac{\mathbf{K}}{\mathrm{f_T^2}} - 2\alpha\frac{\mathrm{f_V}}{\mathrm{f_T^3}}\mathbf{L} + \alpha^2\frac{1}{\mathrm{f_T^3}}\mathbf{L} + \frac{\mathrm{f_V^2}}{\mathrm{f_T^3}}\mathbf{L} \end{pmatrix} \begin{pmatrix} \nabla_0 \mathrm{f_V} \\ \nabla_0 \mathrm{f_T} \end{pmatrix}$$

Fluxes

Coefficient matrix

$\mathbf{Z}$

Field gradients

- $\begin{pmatrix} \mathbf{J}_\mathrm{e} \\ \mathbf{J}_\mathrm{y} \end{pmatrix}$ and $\begin{pmatrix} \nabla_0 \mathrm{f_V} \\ \nabla_0 \mathrm{f_T} \end{pmatrix}$ are conjugated pairs of fluxes and fields gradient

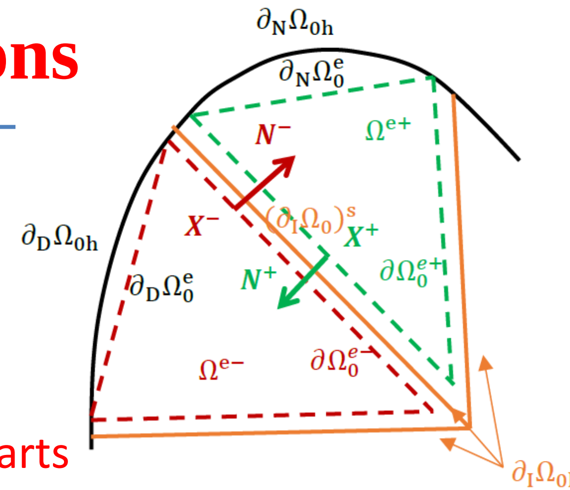
Discontinuous Galerkin (DG) method

- Similarity to FEM, to solve PDE's
  - Geometry approximated by polyhedral elements
  - Continuity ensured inside elements
    - Polynomial solution of finite degree
- Main difference with FEM:
  - Compatibilty weakly ensured
    - Inter-element continuity weakly constrained
    - Support of nodal shape functions restrained to one element

$$\mathrm{X}^{\mathrm{k}(+)} = \left\{ \mathbf{G}_{\mathrm{h}} \in \left[\mathrm{L}^2(\Omega_{0\mathrm{h}})\right]^3 \times \mathrm{L}^2(\Omega_{0\mathrm{h}}) \times \mathrm{L}^{2^{(+)}}(\Omega_{0\mathrm{h}}) \,\middle|_{\mathbf{G}_{\mathrm{h}}|_{\Omega_0^{\mathrm{e}}} \in [\mathbb{P}^{\mathrm{k}}(\Omega_0^{\mathrm{e}})]^3 \times \mathbb{P}^{\mathrm{k}}(\Omega_0^{\mathrm{e}}) \times \mathbb{P}^{\mathrm{k}^{(+)}}(\Omega_0^{\mathrm{e}}) \ \forall \Omega_0^{\mathrm{e}} \in \Omega_{0\mathrm{h}}} \right\}$$

  - Allows / eases:
    - High scalability and high accuracy order
    - Irregular and non-conforming meshes
    - hp-adaptivity

DG main concepts and equations



- Strong form: $\nabla_0 \cdot \mathbf{P}^T = 0, \qquad (+\text{BC's})$
- DG weak form: $\int_{\Omega_{0h}} (\nabla_0 \cdot \mathbf{P}^T) \cdot \delta\mathbf{u} d\Omega_0 = 0$

$\int$ by parts

- Define operators

Jump operator $[\![\mathbf{u}]\!] = \mathbf{u}^+ - \mathbf{u}^-$

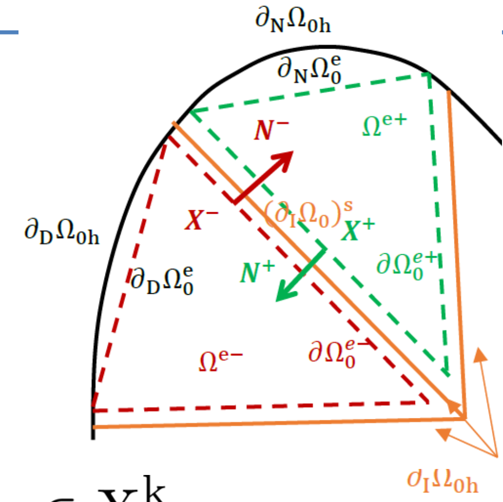
Average operator $\langle \mathbf{u} \rangle = \dfrac{\mathbf{u}^+ + \mathbf{u}^-}{2}$

$$\int_{\Omega_{0h}} \mathbf{P} : \nabla_0 \delta\mathbf{u} d\Omega_0 + \int_{\partial_I\Omega_{0h}} [\![\delta\mathbf{u}]\!] \cdot \langle \mathbf{P} \rangle \cdot \mathbf{N}^- dS_0 + (\ ) + (\ ) = b(\delta\mathbf{u})$$

BC's terms

- Supplementary terms:
  - **Consistency** term (appears naturally above)
  - **Symmetrisation** term (optimal convergence rate) $\int_{\partial_I\Omega_{0h}} [\![\mathbf{u}]\!] \cdot \langle \mathcal{H}_0 : \nabla_0 \delta\mathbf{u} \rangle \cdot \mathbf{N}^- dS_0$
  - Quadratic **stabilization** term ($\mathcal{B}$ = stabilisation parameter) $\int_{\partial_I\Omega_{0h}} [\![\mathbf{u}]\!] \otimes \mathbf{N}^- : \left\langle \dfrac{\mathcal{H}_0 \mathcal{B}}{h_s} \right\rangle : [\![\delta\mathbf{u}]\!] \otimes \mathbf{N}^- dS_0$

Nonlinear DG formulation of Electro-Thermo-Mechanical coupling



- Introducing vector of the unknowns field $\mathbf{G} = \begin{pmatrix} \mathbf{u} \\ \mathrm{f_V} \\ \mathrm{f_T} \end{pmatrix}$
- Defining $\nabla_0(\mathbf{J}) = \nabla_0\mathbb{J}(\mathbf{G}, \nabla_0\mathbf{G}, \mathbb{I}) = \begin{pmatrix} \nabla_0 \cdot \mathbf{P}^{\mathrm{T}} \\ \nabla_0 \cdot \mathbf{J}_{\mathrm{e}} \\ \nabla_0 \cdot \mathbf{J}_{\mathrm{y}} \end{pmatrix} = 0$
- Finding $\mathbf{G}_{\mathrm{h}} \in \mathrm{X}^{\mathrm{k}^+}$

$$\mathrm{A}(\mathbf{G}_{\mathrm{h}}, \delta\mathbf{G}_{\mathrm{h}}) = \mathrm{B}(\bar{\mathbf{G}}, \delta\mathbf{G}_{\mathrm{h}}) \;\; \forall\delta\,\mathbf{G}_{\mathrm{h}} \in \mathrm{X}^{\mathrm{k}}$$

Structural term + DG terms = Boundary terms

$$\mathrm{A}(\mathbf{G}_{\mathrm{h}}, \delta\mathbf{G}_{\mathrm{h}}) = \int_{\Omega_{0\mathrm{h}}} \nabla_0\delta\mathbf{G}_{\mathrm{h}}^{\mathrm{T}}\mathbf{J}(\mathbf{G}_{\mathrm{h}}, \nabla_0\mathbf{G}_{\mathrm{h}})\mathrm{d}\Omega_0 + \int_{\partial_{\mathrm{I}}\Omega_{0\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{0\mathrm{h}}} \left[\!\left[\delta\mathbf{G}_{\mathbf{N}_{\mathrm{h}}}^{\mathrm{T}}\right]\!\right] \langle\mathbf{J}(\mathbf{G}_{\mathrm{h}}, \nabla_0\mathbf{G}_{\mathrm{h}})\rangle\,\mathrm{dS}_0$$
$$+ \int_{\partial_{\mathrm{I}}\Omega_{0\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{0\mathrm{h}}} \left[\!\left[\mathbf{G}_{\mathbf{N}_{\mathrm{h}}}^{\mathrm{T}}\right]\!\right] \langle\mathbf{J}_{0\nabla\mathbf{G}}(\mathbf{G}_{\mathrm{h}})\nabla_0\delta\mathbf{G}_{\mathrm{h}}\rangle\,\mathrm{dS}_0 + \int_{\partial_{\mathrm{I}}\Omega_{0\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{0\mathrm{h}}} \left[\!\left[\delta\mathbf{G}_{\mathbf{N}_{\mathrm{h}}}^{\mathrm{T}}\right]\!\right] \langle\mathbf{J}_{0\mathbf{G}}(\mathbf{G}_{\mathrm{h}})\mathbf{G}_{\mathrm{h}}\rangle\,\mathrm{dS}_0$$
$$+ \int_{\partial_{\mathrm{I}}\Omega_{0\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{0\mathrm{h}}} \left[\!\left[\mathbf{G}_{\mathbf{N}_{\mathrm{h}}}^{\mathrm{T}}\right]\!\right] \left\langle \frac{\mathbf{J}_{0\nabla\mathbf{G}}(\mathbf{G}_{\mathrm{h}})\mathcal{B}}{\mathrm{h_s}} \right\rangle [\![\delta\mathbf{G}_{\mathbf{N}_{\mathrm{h}}}]\!]\,\mathrm{dS}_0$$

Consistency

Symmetry

Stability

$$\mathrm{B}(\bar{\mathbf{G}}, \delta\mathbf{G}_{\mathrm{h}}) = \mathrm{BC}'\mathrm{s}$$

Remark: we use an abuse of notations when defining $\nabla_0\mathbf{G}, \mathbf{N}$

Nonlinear DG formulation of Electro-Thermo-Mechanical coupling

- **Considering small deformation**

$$\nabla\left(\begin{pmatrix}\mathcal{H}_0 & 0 & 0\\ 0 & \mathbf{z}_{11} & \mathbf{z}_{12}\\ 0 & \mathbf{z}_{21} & \mathbf{z}_{22}\end{pmatrix}\begin{pmatrix}\nabla\mathbf{u}\\ \nabla\mathrm{f_V}\\ \nabla\mathrm{f_T}\end{pmatrix}\right)+\begin{pmatrix}0 & 0 & \boldsymbol{\alpha}_{\mathrm{th}}^{\mathrm{T}}\mathcal{H}_0\frac{1}{\mathrm{f_T^2}}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\nabla\mathbf{u}\\ \nabla\mathrm{f_V}\\ \nabla\mathrm{f_T}\end{pmatrix}=0$$

- **Strong form:** $\nabla(\mathbf{w}(\mathbf{G},\nabla\mathbf{G}))+\mathbf{o}(\mathbf{G})\nabla\mathbf{G}=0 \quad \text{in }\Omega. \quad +\mathrm{BC}'\mathrm{s}$

- **Weak form:** $\mathbf{G}_\mathrm{h}\in\mathrm{X}^{\mathrm{k}^+} \qquad \mathrm{a}(\mathbf{G}_\mathrm{h},\delta\mathbf{G}_\mathrm{h})=\mathrm{b}(\bar{\mathbf{G}};\delta\mathbf{G}_\mathrm{h})\ \ \forall\delta\,\mathbf{G}_\mathrm{h}\in\mathrm{X}^\mathrm{k}$

$$\mathrm{a}(\mathbf{G}_\mathrm{h},\delta\mathbf{G}_\mathrm{h})=\int_{\Omega_\mathrm{h}}\nabla\delta\mathbf{G}_\mathrm{h}^\mathrm{T}\mathbf{w}(\mathbf{G}_\mathrm{h},\nabla\mathbf{G}_\mathrm{h})\mathrm{d}\Omega+\int_{\Omega_\mathrm{h}}\mathbf{G}_\mathrm{h}^\mathrm{T}\mathbf{o}(\mathbf{G}_\mathrm{h})\nabla\delta\mathbf{G}_\mathrm{h}\mathrm{d}\Omega$$

$$+\int_{\partial_\mathrm{I}\Omega_\mathrm{h}\cup\partial_\mathrm{D}\Omega_\mathrm{h}}\left[\!\left[\delta\mathbf{G}_{\mathrm{h}_\mathbf{n}}^\mathrm{T}\right]\!\right]\langle\mathbf{w}(\mathbf{G}_\mathrm{h},\nabla\mathbf{G}_\mathrm{h})\rangle\,\mathrm{dS}+\int_{\partial_\mathrm{I}\Omega_{\mathrm{h}\cup\partial_\mathrm{D}\Omega_\mathrm{h}}}\left[\!\left[\mathbf{G}_{\mathrm{h}_\mathbf{n}}^\mathrm{T}\right]\!\right]\langle\mathbf{o}(\mathbf{G}_\mathrm{h})\delta\mathbf{G}_\mathrm{h}\rangle\,\mathrm{dS}$$ → **Consistency**

$$+\int_{\partial_\mathrm{I}\Omega_\mathrm{h}\cup\partial_\mathrm{D}\Omega_\mathrm{h}}\left[\!\left[\mathbf{G}_{\mathrm{h}_\mathbf{n}}^\mathrm{T}\right]\!\right]\langle\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}_\mathrm{h})\nabla\delta\mathbf{G}_\mathrm{h}\rangle\,\mathrm{dS}+\int_{\partial_\mathrm{I}\Omega_{\mathrm{h}\cup\partial_\mathrm{D}\Omega_\mathrm{h}}}\left[\!\left[\delta\mathbf{G}_{\mathrm{h}_\mathbf{n}}^\mathrm{T}\right]\!\right]\langle\mathbf{o}(\mathbf{G}_\mathrm{h})\mathbf{G}_\mathrm{h}\rangle\,\mathrm{dS}$$ → **Symmetry**

$$+\int_{\partial_\mathrm{I}\Omega_\mathrm{h}\cup\partial_\mathrm{D}\Omega_\mathrm{h}}\left[\!\left[\mathbf{G}_{\mathrm{h}_\mathbf{n}}^\mathrm{T}\right]\!\right]\left\langle\frac{\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}_\mathrm{h})\mathcal{B}}{\mathrm{h_s}}\right\rangle[\![\delta\mathbf{G}_{\mathrm{h}_\mathbf{n}}]\!]\,\mathrm{dS}$$ → **Stability**

$$\mathrm{b}(\bar{\mathbf{G}},\delta\mathbf{G}_\mathrm{h})=\mathrm{BC}'\mathrm{s}$$

Solution uniqueness

- **The mesh dependent norm**

$$|||\, \mathbf{G} \,|||_1^2 \quad = \sum\nolimits_e \|\mathbf{G}\|^2_{H^1(\Omega^e)} + \sum\nolimits_s h_s \,\|\, \mathbf{G} \,\|^2_{H^1(\partial\Omega^e)} + \sum\nolimits_s h_s^{-1} \| \, [\![\mathbf{G_n}]\!] \, \|^2_{L^2(\partial\Omega^e)}$$

Where $\partial\Omega^e = \partial_I\Omega^e \cup \partial_D\Omega^e$

- **Consistency form**

$\mathbf{G}^e \in [H^2(\Omega)]^3 \times H^2(\Omega) \times H^{2^+}(\Omega)$ the solution of the strong form.

Thus as $[\![\mathbf{G}^e]\!] = 0$ on $\partial_I\Omega^e$

$$a(\mathbf{G}^e, \delta\mathbf{G}^e) = b(\bar{\mathbf{G}}, \delta\mathbf{G}^e) \;\; \forall \delta\mathbf{G}^e \in X, \qquad (1)$$

- **Weak form**

The weak form, reads as finding $\mathbf{G}_h \in X^k$, such that

$$a(\mathbf{G}_h, \delta\mathbf{G}_h) = b(\bar{\mathbf{G}}; \delta\mathbf{G}_h) \;\; \forall \delta\, \mathbf{G}_h \in X^k \subset X \qquad (2)$$

Replacing $\delta\mathbf{G}^e = \delta\mathbf{G}_h$, then subtracting (2) from (1)

$$a(\mathbf{G}^e, \delta\mathbf{G}_h) - a(\mathbf{G}_h, \delta\mathbf{G}_h) = b(\bar{\mathbf{G}}, \delta\mathbf{G}_h) - b(\bar{\mathbf{G}}, \delta\mathbf{G}_h) = 0 \;\; \forall \delta\mathbf{G}_h \in X^k$$

Solution uniqueness

$$\mathcal{A}(\underline{\mathbf{G}^{\mathrm{e}}};\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}},\delta\mathbf{G}_{\mathrm{h}})+\mathcal{B}(\underline{\mathbf{G}^{\mathrm{e}}};\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}},\delta\mathbf{G}_{\mathrm{h}}) = \mathcal{N}(\mathbf{G}^{\mathrm{e}},\mathbf{G}_{\mathrm{h}};\delta\mathbf{G}_{\mathrm{h}})$$

Fixed **Fixed**

$\mathcal{A}$, $\mathcal{B}$ **Bilinear**

$$\begin{aligned}\mathcal{A}(\mathbf{G}^{\mathrm{e}};\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}},\delta\mathbf{G}_{\mathbf{h}}) = &\int_{\Omega_{\mathrm{h}}}\nabla\delta\mathbf{G}_{\mathrm{h}}^{\mathrm{T}}\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}^{\mathrm{e}})(\nabla\mathbf{G}^{\mathrm{e}}-\nabla\mathbf{G}_{\mathrm{h}})\mathrm{d}\Omega\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}[\![\delta\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\left\langle\mathbf{w}_{\nabla\mathbf{G}}\left(\mathbf{G}^{\mathrm{e}}\right)\left(\nabla\mathbf{G}^{\mathrm{e}}-\nabla\mathbf{G}_{\mathrm{h}}\right)\right\rangle\mathrm{dS}\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}\left[\!\!\left[\mathbf{G}_{\mathbf{n}}^{\mathrm{e}^{\mathrm{T}}}-\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\right]\!\!\right]\left\langle\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}^{\mathrm{e}})\nabla\delta\mathbf{G}_{\mathrm{h}}\right\rangle\mathrm{dS}\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}\left[\!\!\left[\mathbf{G}_{\mathbf{n}}^{\mathrm{e}^{\mathrm{T}}}-\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\right]\!\!\right]\left\langle\frac{\mathcal{B}}{\mathrm{h}_{\mathrm{s}}}\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}^{\mathrm{e}})\right\rangle[\![\delta\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}]\!]\,\mathrm{dS},\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\mathbf{G}^{\mathrm{e}};\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}},\delta\mathbf{G}_{\mathrm{h}}) = &\int_{\Omega_{\mathrm{h}}}\nabla\delta\mathbf{G}_{\mathrm{h}}^{\mathrm{T}}\left(\mathbf{w}_{\mathbf{G}}(\mathbf{G}^{\mathrm{e}},\nabla\mathbf{G}^{\mathrm{e}})\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}}\right)\mathrm{d}\Omega\\ &+\int_{\Omega_{\mathrm{h}}}\nabla\delta\mathbf{G}_{\mathrm{h}}^{\mathrm{T}}\left(\mathbf{o}_{\mathbf{G}}'(\mathbf{G}^{\mathrm{e}})(\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}})\right)\mathrm{d}\Omega\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}[\![\delta\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\left\langle\mathbf{w}_{\mathbf{G}}(\mathbf{G}^{\mathrm{e}},\nabla\mathbf{G}^{\mathrm{e}})(\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}})\right\rangle\mathrm{dS}\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}[\![\delta\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\left\langle\mathbf{o}_{\mathbf{G}}'(\mathbf{G}^{\mathrm{e}})(\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}})\right\rangle\mathrm{dS}\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}\left[\!\!\left[\mathbf{G}_{\mathbf{n}}^{\mathrm{e}^{\mathrm{T}}}-\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\right]\!\!\right]\left\langle\mathbf{o}(\mathbf{G}^{\mathrm{e}})\delta\mathbf{G}_{\mathrm{h}}\right\rangle\mathrm{dS}.\end{aligned}$$

Expansion of Taylor series on interface terms

$$\begin{aligned}\mathcal{N}(\mathbf{G}^{\mathrm{e}},\mathbf{G}_{\mathrm{h}};\delta\mathbf{G}_{\mathrm{h}}) = &\int_{\Omega_{\mathrm{h}}}\nabla\delta\mathbf{G}_{\mathrm{h}}^{\mathrm{T}}(\bar{\mathbf{R}}_{\mathbf{w}}(\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}},\nabla\mathbf{G}^{\mathrm{e}}-\nabla\mathbf{G}_{\mathrm{h}}))\mathrm{d}\Omega\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}[\![\delta\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\left\langle\bar{\mathbf{R}}_{\mathbf{w}}(\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}},\nabla\mathbf{G}^{\mathrm{e}}-\nabla\mathbf{G}_{\mathrm{h}})\right\rangle\mathrm{dS}\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}\left[\!\!\left[\mathbf{G}_{\mathbf{n}}^{\mathrm{e}^{\mathrm{T}}}-\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\right]\!\!\right]\left\langle\left(\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}^{\mathrm{e}})-\mathbf{v}_{\nabla\mathbf{G}}(\mathbf{G}_{\mathrm{h}})\right)\nabla\delta\mathbf{G}_{\mathrm{h}}\right\rangle\mathrm{dS}\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}\left[\!\!\left[\mathbf{G}_{\mathbf{n}}^{\mathrm{e}^{\mathrm{T}}}-\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\right]\!\!\right]\left\langle\frac{\mathcal{B}}{\mathrm{h}_{\mathrm{s}}}\left(\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}^{\mathrm{e}})-\mathbf{w}_{\nabla\mathbf{G}}(\mathbf{G}_{\mathrm{h}})\right)\right\rangle[\![\delta\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}]\!]\,\mathrm{dS}\\ &-\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}\left[\!\!\left[\mathbf{G}_{\mathbf{n}}^{\mathrm{e}^{\mathrm{T}}}-\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\right]\!\!\right]\left\langle\left(\mathbf{o}(\mathbf{G}^{\mathrm{e}})-\mathbf{o}(\mathbf{G}_{\mathrm{h}})\right)\delta\mathbf{G}_{\mathrm{h}}\right\rangle\mathrm{dS}\\ &+\int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}\cup\partial_{\mathrm{D}}\Omega_{\mathrm{h}}}[\![\delta\mathbf{G}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\left\langle\bar{\mathbf{R}}_{\mathbf{G}}(\mathbf{G}^{\mathrm{e}}-\mathbf{G}_{\mathrm{h}})\right\rangle\mathrm{dS}\end{aligned}$$

Nonlinear

Solution uniqueness

Spliting ζ into its components ⇒

$$\zeta = \mathbf{G}^{\mathrm{e}} - \mathbf{G}_{\mathrm{h}} - \mathrm{I}_{\mathrm{h}}\mathbf{G} + \mathrm{I}_{\mathrm{h}}\mathbf{G}$$
$$= \boldsymbol{\eta} + \boldsymbol{\xi}$$

$\mathrm{I}_{\mathrm{h}}\mathbf{G}$: The interpolant of $\mathbf{G}^{\mathrm{e}}$ in X^{k}

$$\boldsymbol{\eta} = \mathbf{G}^{\mathrm{e}} - \mathrm{I}_{\mathrm{h}}\mathbf{G} \in \mathrm{X} \qquad \boldsymbol{\xi} = \mathrm{I}_{\mathrm{h}}\mathbf{G} - \mathbf{G}_{\mathrm{h}} \in \mathrm{X}^{\mathrm{k}}$$

$$\mathcal{A}(\mathbf{G}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{G} - \mathbf{G}_{\mathrm{h}}, \delta\mathbf{G}_{\mathrm{h}}) + \mathcal{B}(\mathbf{G}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{G} - \mathbf{G}_{\mathrm{h}}, \delta\mathbf{G}_{\mathrm{h}})$$
$$= \mathcal{A}(\mathbf{G}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{G}_{\mathrm{h}}) + \mathcal{B}(\mathbf{G}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{G}_{\mathrm{h}}) + \mathcal{N}(\mathbf{G}^{\mathrm{e}}, \mathbf{G}_{\mathrm{h}}; \delta\mathbf{G}_{\mathrm{h}})$$

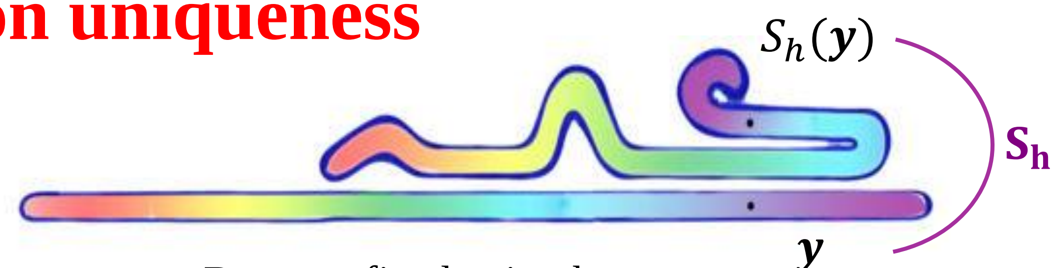
Fixed point formulation

Map $\mathrm{S}_{\mathrm{h}} : \mathrm{X}^{\mathrm{k}} \to \mathrm{X}^{\mathrm{k}}$ as follows

$$\forall \mathbf{y} \in \mathrm{X}^{\mathrm{k}}, \text{ Find } \mathrm{S}_{\mathrm{h}}(\mathbf{y}) = \mathbf{G}_{\mathbf{y}} \in \mathrm{X}^{\mathrm{k}}$$

$$\mathcal{A}(\mathbf{G}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{G} - \mathbf{G}_{\mathbf{y}}, \delta\mathbf{G}_{\mathrm{h}}) + \mathcal{B}(\mathbf{G}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{G} - \mathbf{G}_{\mathbf{y}}, \delta\mathbf{G}_{\mathrm{h}})$$
$$= \mathcal{A}(\mathbf{G}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{G}_{\mathrm{h}}) + \mathcal{B}(\mathbf{G}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{G}_{\mathrm{h}}) + \mathcal{N}(\mathbf{G}^{\mathrm{e}}, \mathbf{y}; \delta\mathbf{G}_{\mathrm{h}}) \quad (*)$$

Solution uniqueness



Brouwer fixed point theorem

The existence of the discrete solution G_h

=

The existence of a fixed point $S_h(y) = y$ in the map S_h

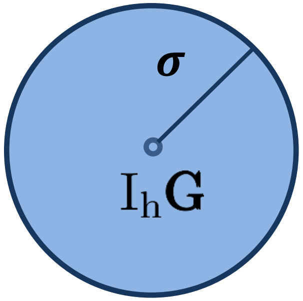
- **S_h map to itself**
- **S_h is continuous map**

T. Gudi, N. Nataraj, A. K. Pani, Numer. Math. 2008

Solution uniqueness

Definition of the ball O_σ

- Radius: σ
- Center: $I_h\mathbf{G}$ the interpolant of $\mathbf{G}^e$



$$O_\sigma(I_h\mathbf{G}) = \left\{ \mathbf{y} \in X^k \text{ such that } |||\, I_h\mathbf{G} - \mathbf{y} \,|||_1 \leq \sigma \right\}$$

$$\text{with} \quad \sigma = \frac{|||\, I_h\mathbf{G} - \mathbf{G}^e \,|||_1}{h_s^\varepsilon}, \quad 0 < \varepsilon < \frac{1}{4}$$

Solution uniqueness

- Assumption C_α, C_y, C^k and Lemmas (e.g. trace inequality, inverse inequality)
- Bound the bilinear terms $\mathcal{A}$, $\mathcal{B}$
- Bound the nonlinear term $\mathcal{N}$

Stabilization parameter β >Const (C_α, C_y, C^k..)

S_h maps $O_\sigma(I_h\mathbf{G})$ into itself

Continuity of S_h in the ball $O_\sigma(I_h\mathbf{G})$

$h_s \longrightarrow 0 \Rightarrow I_h\mathbf{G} - \mathbf{G_y} \longrightarrow 0$

$||| \mathbf{G}_{\mathbf{y}_1} - \mathbf{G}_{\mathbf{y}_2} ||| \leq C^k h_s^{\mu-2-\varepsilon} ||| \mathbf{y}_1 - \mathbf{y}_2 |||$

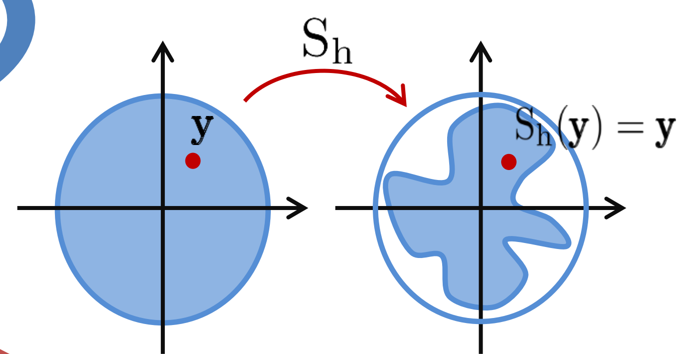
$\mathbf{y} \in O_\sigma(I_h\mathbf{G})$

$S_h(\mathbf{y}) = \mathbf{y}$

Brouwer fixed point

$S_h(\mathbf{y})$ **has a fixed point** $\mathbf{G_h}$

The existence of unique solution of the nonlinear elliptic problem for k ≥2



Brouwer fixed point theorem

A prior error estimates

H^1-norm

$$|||\, \mathbf{G}^e - \mathbf{G}_h \,|||_1 \quad \leq C^k h_s^{\mu-1} \,\|\, \mathbf{G}^e \,\|_{H^s(\Omega_h)}$$

$$\mu = \min\{s, k+1\}$$

L^2-norm

$$\|\, \mathbf{G}^e - \mathbf{G}_h \,\|_{L^2(\Omega_h)} \leq C^k h_s^{\mu} \,\|\, \mathbf{G}^e \,\|_{H^s(\Omega_h)}$$

H^1, L^2-norms are optimal in the mesh size for <u>linear</u> elliptic problem

=

H^1, L^2-norms are optimal in the mesh size for <u>nonlinear</u> elliptic problem

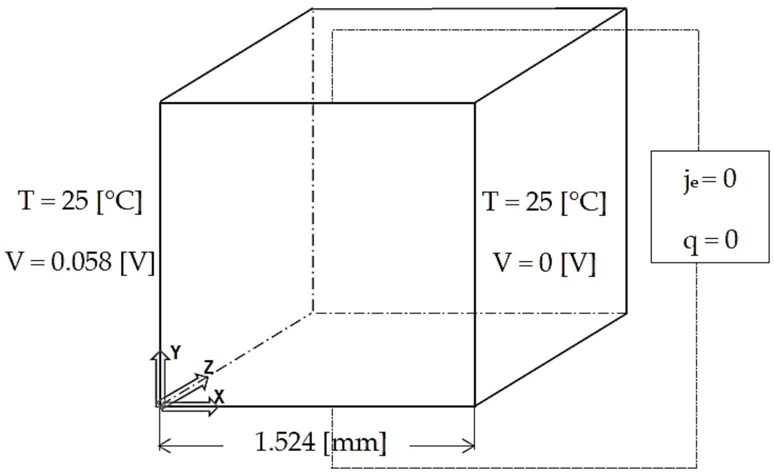
Numerical results
1-D example with one material

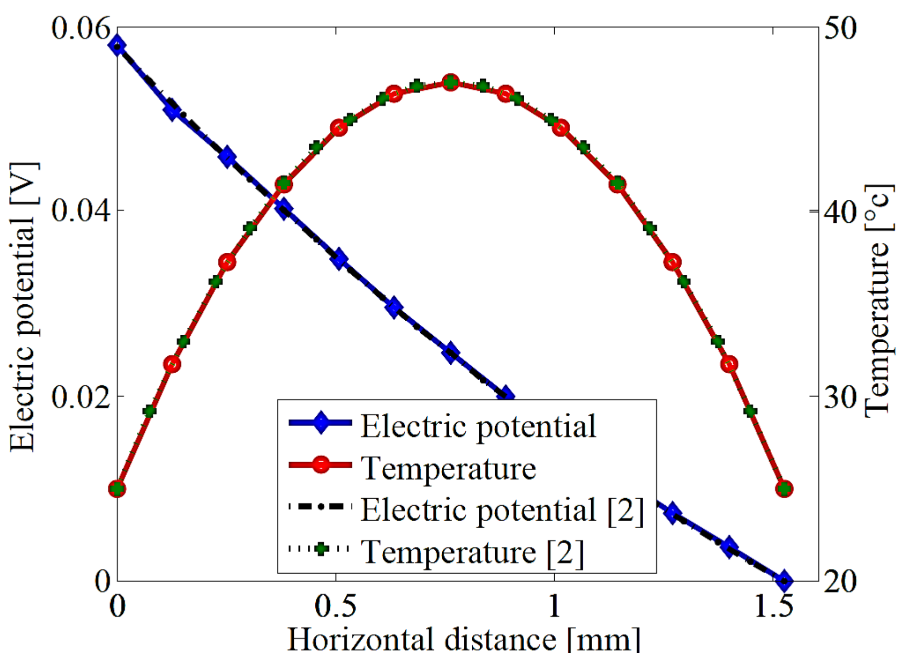
(Electro-Thermal coupling)

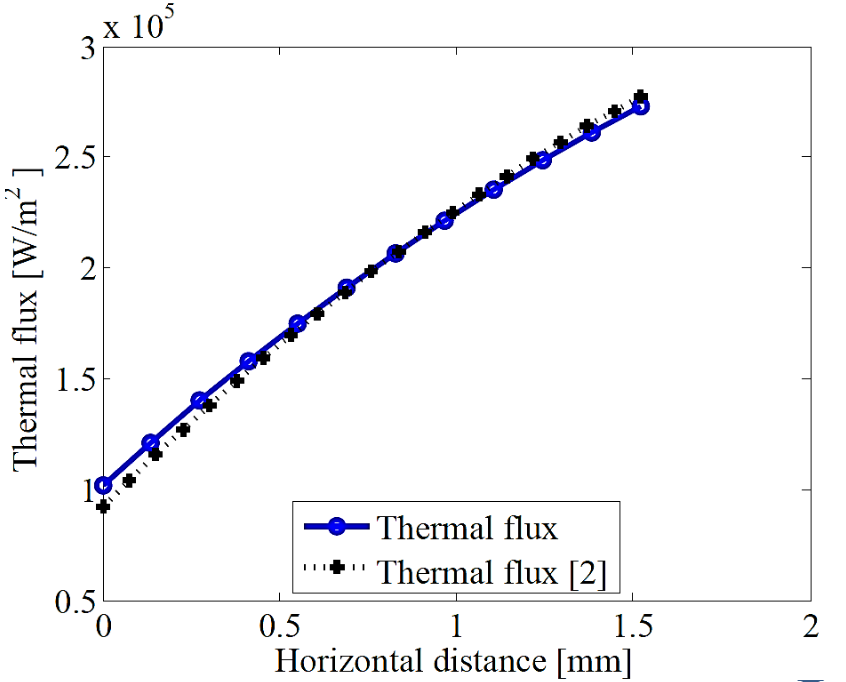
Material parameters of bismuth telluride

$\mathbf{l}$ [S/m]	$\mathbf{k}$ [W/(K · m)]	α [V/K]
diag(8.422×10^4)	diag(1.612)	1.941×10^{-4}

[2]. L. Liu. International Journal of Engineering Science, 2012





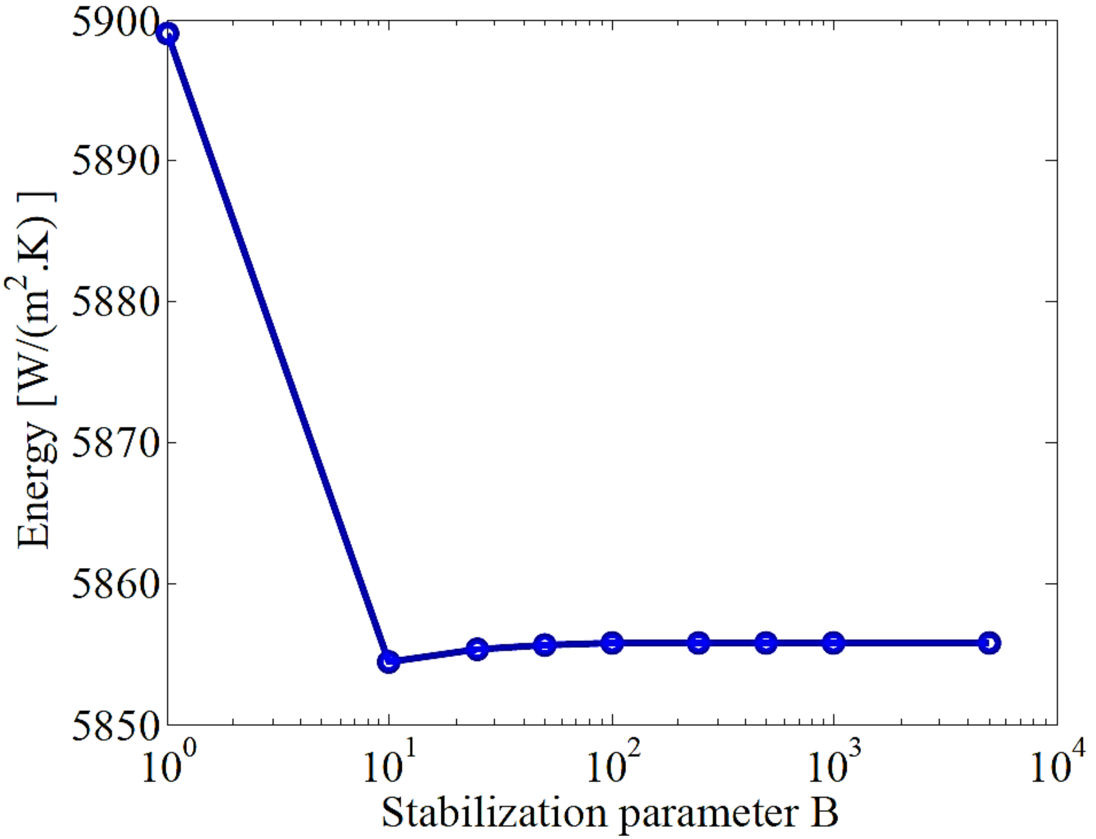


Numerical results

1-D example with two materials

(Electro-Thermal coupling)

The effect of the **stabilization parameter** on the quality of the approximation

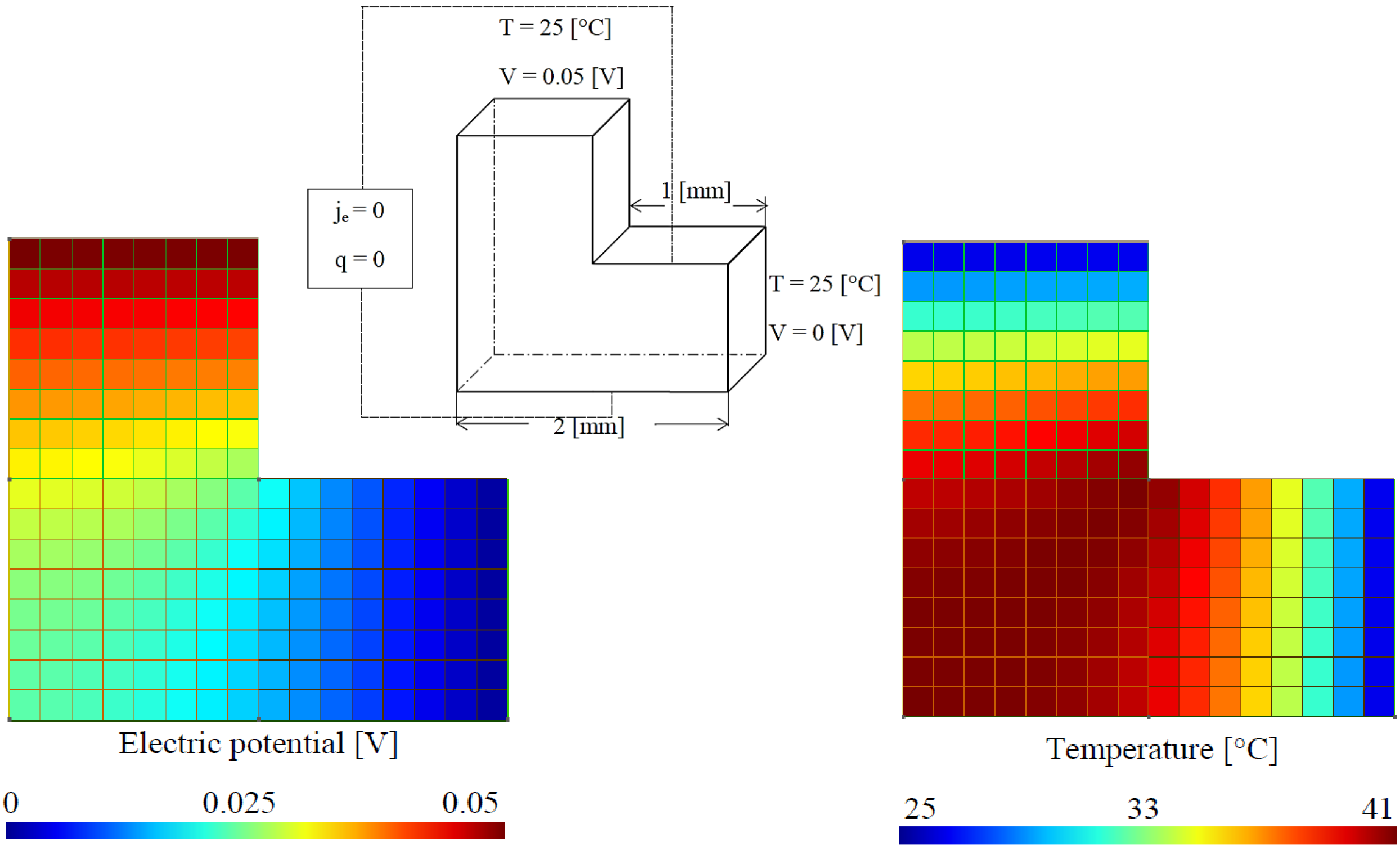


DG formulation is stable for stabilization parameter >10

Numerical results

2-D study of convergence order

(Electro-Thermal coupling)

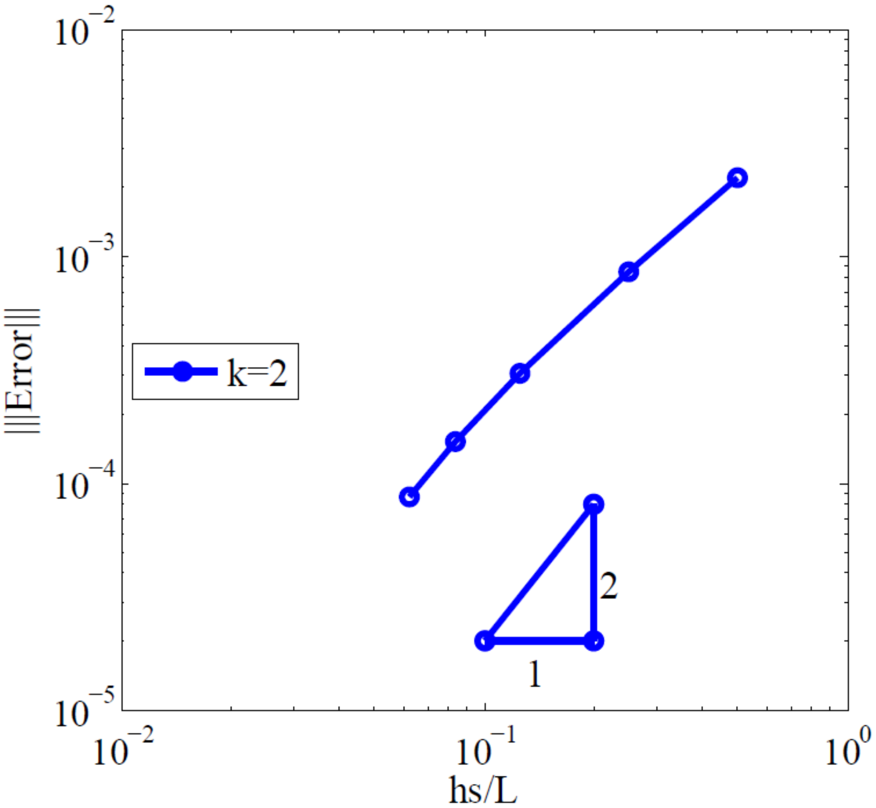


Numerical results
2-D study of convergence order

(Electro-Thermal coupling)

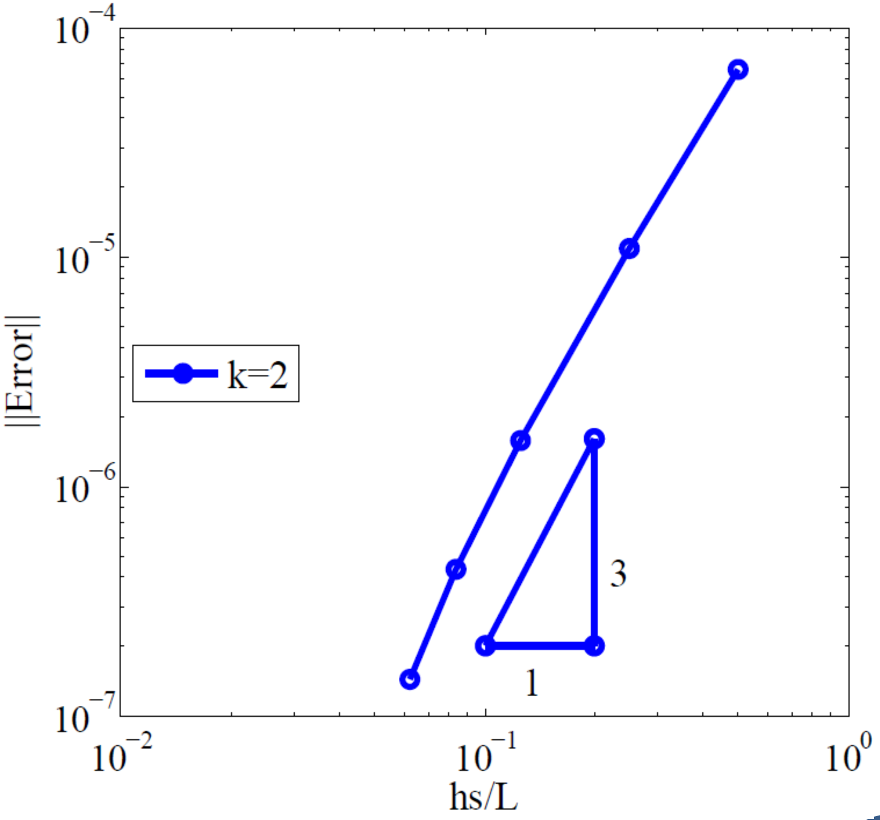
H^1-norm

Theor. converg. ord.: k



L^2-norm

Theor. converg. ord.: k+1



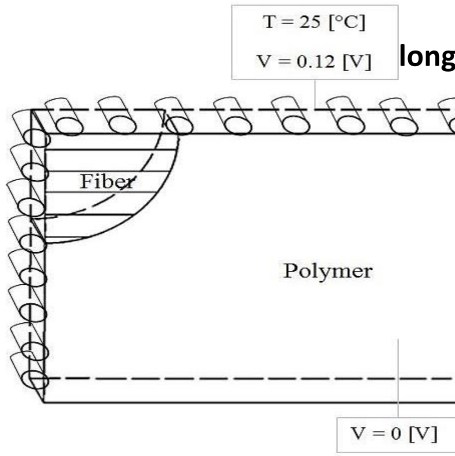
Convergence rates agree with the theoretical estimates

Numerical results

3-D unit cell simulation for composite material

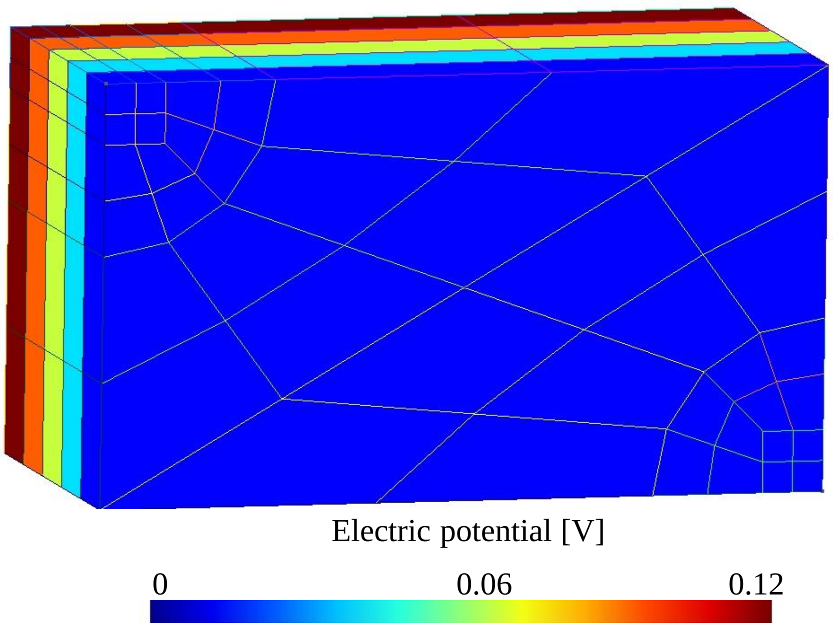
(Electro-Thermo-Mechanical coupling)

Material	$\mathbf{l}$ [S/m]	$\mathbf{k}$ [W/(K · m)]	α [V/K]	$\boldsymbol{\alpha}_{th}$ [K^{-1}]	E_L [GPa]	E_T [GPa]
Carbon fiber	diag(100000)	diag(40)	3×10^{-6}	diag(2×10^{-6})	230	40
Polymer	diag(0.1)	diag(0.2)	3×10^{-7}	diag(20×10^{-5})	1.5	1.5

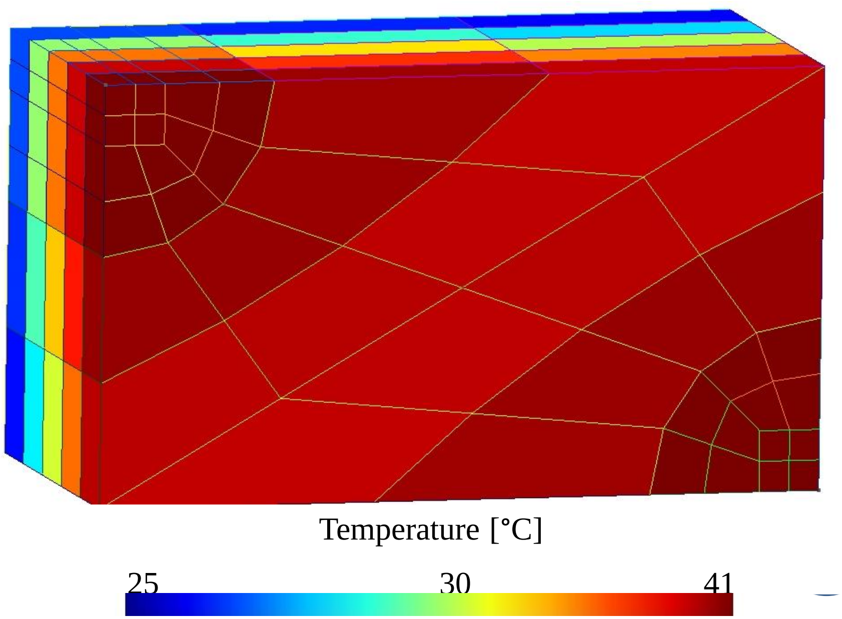


BC on longitudinal direction

DG formulation is also applicable for irregular mesh



Electric potential [V]

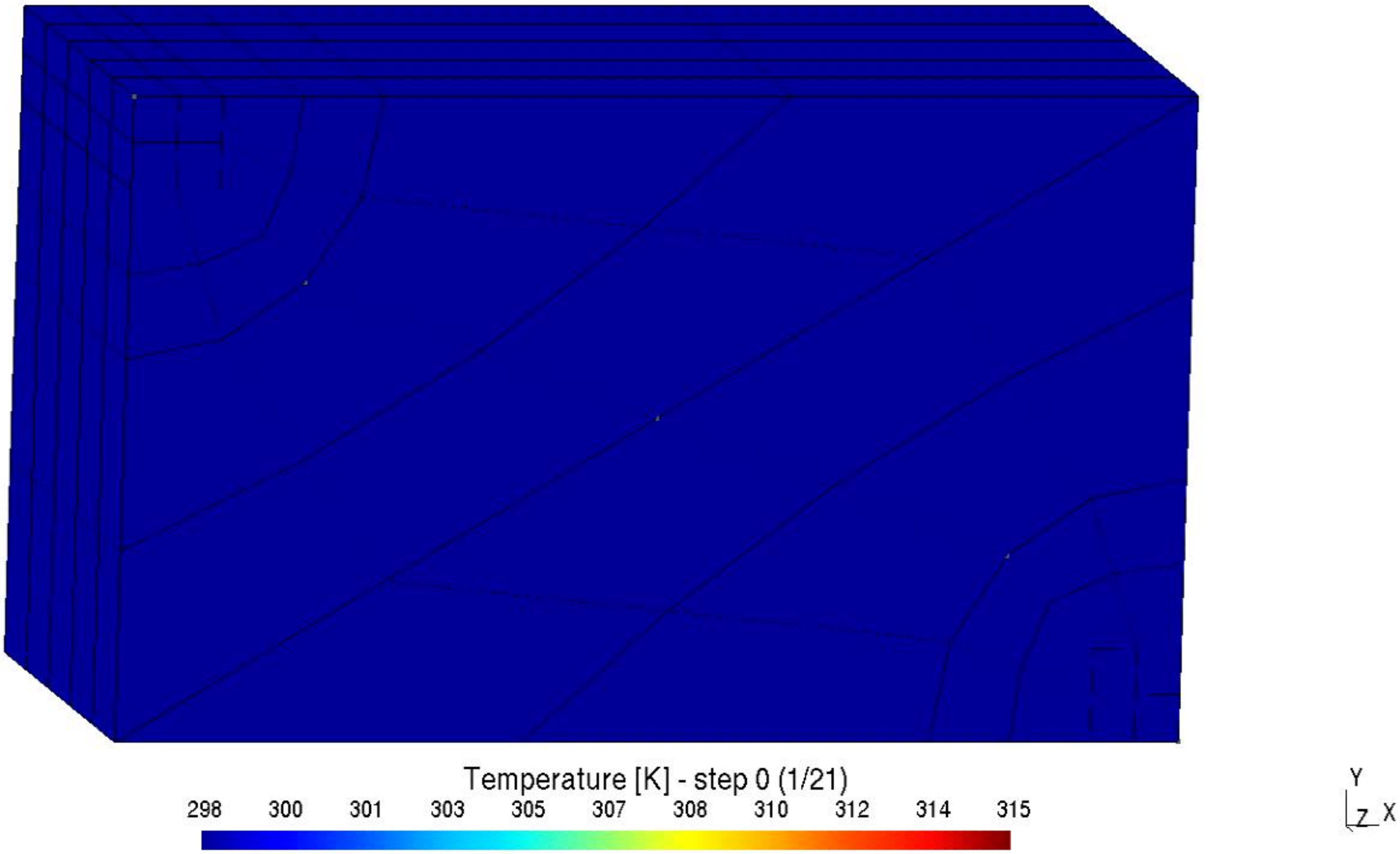


Temperature [°C]

Numerical results

3-D unit cell simulation for composite material

(Electro-Thermo-Mechanical coupling)



magnified

Numerical results

3-D unit cell simulation for composite material

(Electro-Thermo-Mechanical coupling)

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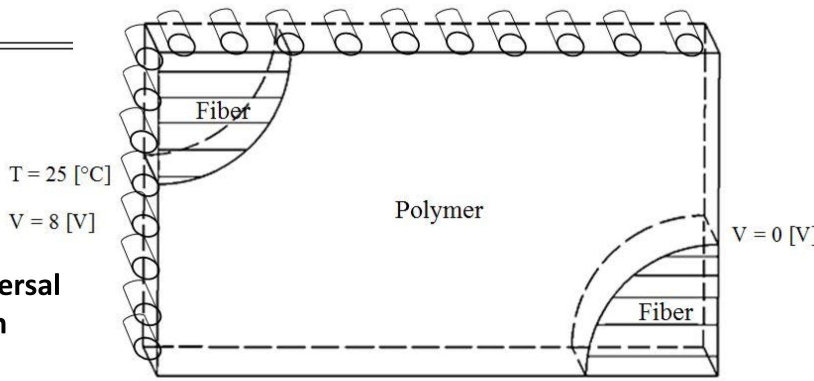
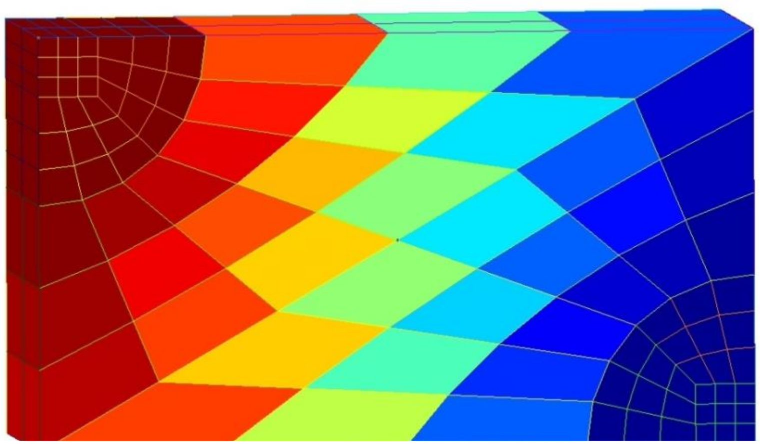
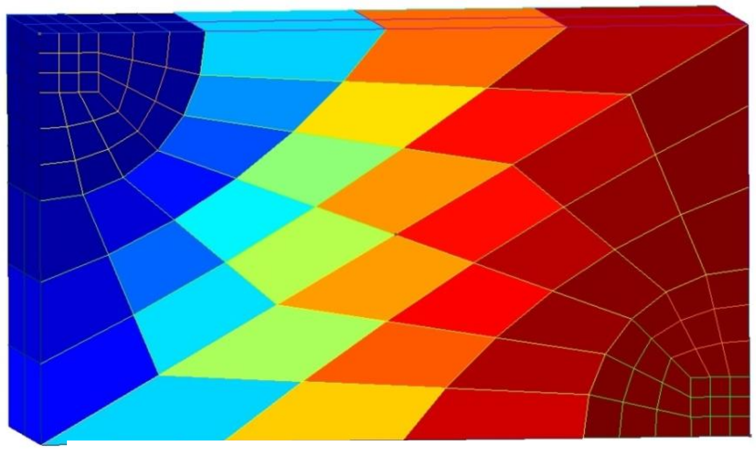
DG formulation is also applicable for irregular mesh



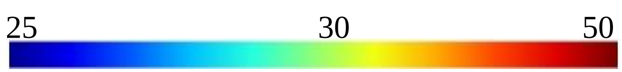
BC on tansversal direction



Electric potential [V]



Temperature [°C]



Conclusion & Perspectives

Conclusion

- A consistent and stable DG method was developed for Electro-Thermo-Mechanical coupled problems
- The DG numerical properties were derived:
  - Uniqueness fixed point form
  - Optimal convergence rates in L_2, H_1-norm with respect to the mesh size
  - Convergence rates agree with the error analysis derived in the theory

Perspectives

- Extension to Electro-Thermo-Mechanical coupled problems to recover shape memory composite material behavior
- Plug in multiscale analyses

Thank you for your attention 🙂

Lina.Homsi@student.ulg.ac.be

Université de Liège