A stroll along the gamma

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Abstract

We provide the first in-depth study of the "smart path" interpolation between an arbitrary probability measure and the gamma-(α, λ) distribution. We propose new explicit representation formulae for the ensuing process as well as a new notion of relative Fisher information with a gamma target distribution. We use these results to prove a differential and an integrated De Bruijn identity which hold under minimal conditions, hereby extending the classical formulae which follow from Bakry, Emery and Ledoux's Γ-calculus. Exploiting a specific representation of the "smart path", we obtain a new proof of the logarithmic Sobolev inequality for the gamma law with $\alpha \geq 1/2$ as well as a new type of HSI inequality linking relative entropy, Stein discrepancy and standardized Fisher information for the gamma law with $\alpha \geq 1/2$.

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1   Introduction

Several choices of metrics are possible in order to quantify the distance between two probability measures, $\mathbb{P}$ and $\mathbb{Q}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A natural choice is the so-called total variation distance between $\mathbb{P}$ and $\mathbb{Q}$ defined by:

$$\mathrm{TV}(\mathbb{P}, \mathbb{Q}) = \sup_{A \in \mathcal{B}(\mathbb{R})} \big|\mathbb{P}(A) - \mathbb{Q}(A)\big|.$$

Assume now that $\mathbb{P}$ and $\mathbb{Q}$ are absolutely continuous with respect to the Lebesgue measure, with densities f and g respectively. Then, a strong control of the total variation distance between $\mathbb{P}$ and $\mathbb{Q}$ can be achieved by means of the well-known Pinsker inequality:

$$\mathrm{TV}(\mathbb{P}, \mathbb{Q}) \leq \sqrt{\frac{D(f\|g)}{2}},$$

where

$$D(f\|g) = \int_{\mathbb{R}} f(x) \log\left(\frac{f(x)}{g(x)}\right) dx$$

is the relative entropy of f with respect to g (or equivalently of $\mathbb{P}$ with respect to $\mathbb{Q}$). Controlling efficiently the relative entropy $D(f\|g)$ in order to quantify the distance between $\mathbb{P}$ and $\mathbb{Q}$ is a natural

strategy which has been thoroughly addressed when $\mathbb{Q}$ is the standard Gaussian probability measure on $\mathbb{R}$ (or on $\mathbb{R}^n$ as well),

$$\mathbb{Q}(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx.$$

More precisely, many authors have provided proofs of quantitative versions of the central limit theorem by means of relative entropy and other information-theoretic quantities. This line of research has its roots in Linnik's works ([26]), culminated with the works of Artstein, Ball, Barthe and Naor in [1, 2], of Johnson and Barron in [24] and of Tulino and Verdu in [36] and to this date is still an active and fruitful area of research [9, 10, 11].

Controlling the relative entropy is by no means easier than the total variation and, at the core of all the previously cited literature, stands the celebrated *De Bruijn formula* linking the relative entropy with one further ingredient: the Fisher information. Namely, for any real random variable X with mean 0, unit variance and whose law $\mathbb{P} = \mathbb{P}_X$ is absolutely continuous:

$$D(\mathbb{P}_X \| \mathbb{Q}) = \int_0^1 \frac{J_{st}(\sqrt{t}X + \sqrt{1-t}Z)}{2t} dt = \int_0^1 \frac{I(\sqrt{t}X + \sqrt{1-t}Z) - 1}{2t} dt, \tag{1}$$

where Z is an independent standard normal random variable, $J_{st}(Y) = \mathbb{E}[(\rho_Y(Y)+Y)^2]$ is the relative Fisher information of Y, $I(Y) = E[\rho_Y(Y)^2]$ is its Fisher information and $\rho_Y(u) = \partial/\partial_u(\log(f_Y(u)))$ is the score function of Y. As discussed in [15], identity (1) holds solely under a second moment assumption on the density of X and was first derived and exploited in Barron ([6, 7]). The first occurrence of such an identity can be traced back to [4, 5] albeit this time under quite stringent regularity assumptions on the density of X. De Bruijn's formula (1) is also of central importance for the proof of functional inequalities such as the logarithmic Sobolev inequality and the more recent HSI inequality (see [22, 4, 14, 25, 12] and [5] for a modern account on this and related topics).

Gamma limit theorems have been obtained in the context of Stein's method on Wiener chaos (see [29, 30]). Influenced by the recent paper [32], it is therefore natural to enquire whether entropic gamma limit theorems on Wiener chaos can be obtained as in the Gaussian case. A first obstacle to a positive answer to this question is the validity of a general De Bruijn-type formula such as (1) in the gamma case.

To our knowledge, De Bruijn-type formulae with general reference probability measure are only known in the context of Γ-calculus (see [4, 5]). Let us inspect how (1) translates when the reference measure is the gamma law. For this purpose, we introduce some notations. We denote by $\gamma_{\alpha,\lambda}(.)$ the density of the gamma law of parameters (α, λ) given by:

$$\forall u > 0, \ \gamma_{\alpha,\lambda}(u) = \frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} \exp(-\lambda u).$$

Let $P_t^{(\alpha,\lambda)}$, $\mathcal{L}_{\alpha,\lambda}$ and $\Gamma_{\alpha,\lambda}$ be the Laguerre semigroup, the Laguerre operator and the carré du champs operator naturally associated with the gamma probability measure (which is invariant and reversible under the Laguerre dynamic). These operators are given respectively by the following formulae on subsets of $L^2(\gamma_{\alpha,\lambda}(u)du)$ (see [21]):

$$P_t^{(\alpha,\lambda)}(f)(x) = \frac{\lambda^{1-\alpha}\Gamma(\alpha)e^{\lambda t}}{e^{\lambda t} - 1} \int_0^{+\infty} f(u) \left(\frac{e^{\lambda t}}{xu}\right)^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda}{e^{\lambda t}-1}(x+u)\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{uxe^{\lambda t}}}{e^{\lambda t}-1}\right) \gamma_{\alpha,\lambda}(du),$$

$$\mathcal{L}_{\alpha,\lambda}(f)(u) = u\frac{d^2 f}{du^2}(u) + (\alpha - \lambda u)\frac{df}{du}(u),$$

$$\Gamma_{\alpha,\lambda}(f)(u) = u\left(\frac{df}{du}(u)\right)^2,$$

where $I_{\alpha-1}(.)$ is the modified Bessel function of the first kind of order $\alpha - 1$. We can now write the local form of De Bruijn identity with respect to the gamma measure as stated in [5, Proposition 5.2.2]:

$$\forall f \geq 0, \ f \in \mathcal{D}(\mathcal{E}), \ \frac{d}{dt}\text{Ent}_{\gamma_{\alpha,\lambda}}(P_t^{(\alpha,\lambda)}(f)) = -I_{\gamma_{\alpha,\lambda}}(P_t^{(\alpha,\lambda)}(f)), \tag{2}$$

2

with,

$$\text{Ent}_{\gamma_{\alpha,\lambda}}(f) = \int_0^{+\infty} f(x) \log(f(x)) \gamma_{\alpha,\lambda}(x) dx, \ I_{\gamma_{\alpha,\lambda}}(f) = \int_0^{+\infty} \frac{\Gamma_{\alpha,\lambda}(f)(x)}{f(x)} \gamma_{\alpha,\lambda}(x) dx,$$

and

$$f \in \mathcal{D}(\mathcal{E}) \Leftrightarrow \int_0^{+\infty} \Gamma_{\alpha,\lambda}(f)(x) \gamma_{\alpha,\lambda}(x) dx < +\infty.$$

When $f = f_X/\gamma_{\alpha,\lambda}$ with f_X the Lebesgue density of a probability measure on $(0, +\infty)$, $\text{Ent}_{\gamma_{\alpha,\lambda}}(f) = D(\mathbb{P}_X \| \gamma_{\alpha,\lambda})$ leading to a gamma-(α, λ)-specific De Bruijn identity:

$$D(X \| \gamma_{\alpha,\lambda}) = \int_0^\infty J_{st,\gamma}(X_t) dt \tag{3}$$

with X_t having Lebesgue density $\gamma_{\alpha,\lambda}(u) P_t^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})(u)$ and $J_{st,\gamma}(Y) = E[Y(\rho_Y(Y) + \lambda - (\alpha - 1)/Y)^2]$ the resulting relative Fisher information (see Section 3 for explanations).

This semigroup approach is not reserved towards a gamma target probability measure and can be adapted e.g. to invariant measures of second order differential operators, which also includes beta distributions as well as families of log-concave measures as illustrations. Identity (3) is, however, not as satisfactory as the Gaussian de Bruijn identity (1) for two reasons.

1. First, if X is a Wiener chaos random variable, the condition $f_X/\gamma_{\alpha,\lambda} \in \mathcal{D}(\mathcal{E})$ is not testable since very little information regarding this type of random variable is available in the literature (see [23, 34, 31]). Moreover, this regularity assumption upon the law of X is a tad uncanny compared to the Gaussian case where only moment conditions need to hold to ensure the validity of the local form of the Gaussian De Bruijn identity (see [7, 6, 15]).

2. Second, and perhaps more importantly, there lacks an explicit representation of X_t in terms of the contributions X and $\gamma_{\alpha,\lambda}$; such a representation (in the Gaussian case $X_t = \sqrt{t}X + \sqrt{1-t}Z$) is indeed at the heart of all above cited literature and its absence in the gamma case dashes any hopes of developing similar techniques for a gamma target as those developed in the Gaussian case.

The main results of this paper address these two issues : in Section 2 we present explicit stochastic representations of the interpolation scheme between the probability measures $\mathbb{P}_X$ and $\gamma_{\alpha,\lambda}$ along the Laguerre dynamic under minimal assumptions on $\mathbb{P}_X$ (hereby solving point 2) which we then use in Section 4 to overcome the regularity conditions necessary for the local De Bruijn identity (hereby solving point 1). In particular we obtain very general forms of local and integrated De Bruijn identity in the gamma case (see Theorem 13 and Remark 14 and Theorem 15 and Remark 16, respectively). As applications we obtain a new proof of the logarithmic Sobolev inequality in the case $\alpha \geq 1/2$ (Proposition 18, Section 5) as well as a new form of HSI inequality for the gamma measure with $\alpha \geq 1/2$ (Theorem 23, Section 6).

To enhance the readability of the text, the next Subsection 1.1 contains an intuitive description of our method and of our main results, as well as more insight into the difference between our results and those already available from the literature (with particular emphasis on a comparison with the well-trodden Gaussian case and formula (1)).

1.1 Overview of the results

Let X be a positive random variable. The crucial first step of this article is to identify the correct probabilistic representation for the smart path evolute X_t from formula (3). Our construction is a priori not intuitive, as it relies on an interpolation between the target gamma distribution and a random variable $Y = Y(X)$ defined according to the following three stage-procedure. Fix $\tau \in (0, 1)$. First draw a (strictly positive) real number, x, according to $\mathbb{P}_X$; next draw a Poisson random variable

with parameter $x\lambda\tau/(1-\tau)$. Finally, depending on the outcome, draw a gamma random variable with shape parameter equal to the value of the Poisson random variable (with the convention that a gamma random variable with shape parameter equal to 0 is 0 almost surely) and scale parameter equal to $\lambda\tau/(1-\tau)$. We denote $Y(\tau, X, \lambda)$ the resulting random variable and define

$$X_\tau = (1-\tau)\gamma(\alpha,\lambda) + \tau Y(\tau, X, \lambda), \tag{4}$$

with $\gamma(\alpha,\lambda)$ an independent gamma random variable with density $\gamma_{\alpha,\lambda}$ (see Definition 1 for a more formal construction).

In Section 2 we will show that X_τ as defined by (4) is the gamma-(α,λ) equivalent of the Gaussian (Ornstein-Uhlenbeck) interpolation $X_t = \sqrt{t}X + \sqrt{1-t}Z$. Note in particular how, in (4), the contributions of X and $\gamma(\alpha,\lambda)$ are dissociated, as is desirable for applications. Moreover, when $\mathbb{P}_X$ admits a Lebesgue density, f_X, the random variable X_τ is linked to the Laguerre dynamic through the following formula for its density $g^{(\alpha,\lambda)}(\tau, u)$:

$$g^{(\alpha,\lambda)}(\tau, u) = \gamma_{\alpha,\lambda}(u) P^{(\alpha,\lambda)}_{-\frac{\log(\tau)}{\lambda}}\big(f_X/\gamma_{\alpha,\lambda}\big)(u).$$

The construction of the random variable X_τ and the existence of $g^{(\alpha,\lambda)}(\tau, u)$ do not rely on the existence of a Lebesgue density for the initial probability measure, $\mathbb{P}_X$. Working directly with $D(X_\tau\|\gamma_{\alpha,\lambda})$ will lead to a substantial improvement of the local form of De Bruijn identity (2), as will be proved in Theorem 13, Section 4, where we obtain an equivalent identity which holds true solely under a moment condition for $\mathbb{P}_X$ whether a density exists or not for the random variable X.

The relative Fisher information we identify for the gamma target distribution is the same as that obtained through the semigroup approach in (3), namely $J_{st,\gamma}(Y) = E[Y(\rho_Y(Y) + \lambda - (\alpha-1)/Y)^2]$, with $\rho_Y(u)$ the classical score of Y. Still using the notation $\gamma(\alpha,\lambda)$ for a gamma-(α,λ) distributed random variable we note how $\rho_{\gamma(\alpha,\lambda)}(u) = (\alpha-1)/u - \lambda$ so that the relative Fisher information can equivalently be rewritten $J_{st,\gamma}(Y) = E[Y(\rho_Y(Y) - \rho_{\gamma(\alpha,\lambda)}(Y))^2]$. When $\alpha > 1$ one can simply expound the square and apply integration by parts to obtain

$$J_{st,\gamma}(Y) = E[Y\rho_Y(Y)^2] - E[Y\rho_{\gamma(\alpha,\lambda)}(Y)^2]$$

which holds as long as $E[Y] = \alpha/\lambda$ and $E[1/Y] = \lambda/(\alpha-1)$. In view of this simple and intuitive computation it might be tempting to introduce a new "gamma-Fisher information" of the form $I_Y(Y) = E[Y\rho_Y(Y)^2]$ for which the above computation leads to the elegant fact that the relative Fisher information decomposes into the difference of gamma-Fisher informations, similarly as in the Gaussian case. The quantity $I_Y(Y)$, however, suffers many flaws, the most cumbersome of which being that it is only finite if $\alpha > 1$ if $Y \sim \gamma_{\alpha,\lambda}$. Such an assumption is not natural. Interestingly, along the course of the proofs of Theorems 13 and 15 (integrated version of De Bruijn identity), we will be led to introducing a new notion of Fisher information (Definition 2):

$$I_\gamma^\tau(Y) = \mathbb{E}\big[Y\big(\rho_Y(Y) + \frac{\lambda}{(1-\tau)} - \frac{\alpha-1}{Y}\big)^2\big].$$

Note how $I_\gamma^\tau(\gamma(\alpha,\lambda)) = \alpha\lambda\tau^2/(1-\tau)^2$ is finite for every value of $\alpha > 0$. We will also prove (Proposition 10) that this information satisfies the Cramer-Rao inequality

$$J_{st,\gamma}(X_\tau) = I_\gamma^\tau(X_\tau) - \frac{\alpha\lambda\tau^2}{(1-\tau)^2} \geq 0.$$

at all points τ along the smart path (see Section 3, Propositions 10 and 11) when $\mathbb{E}[X] = \alpha/\lambda$. This corrected and localized version of Fisher information seems to be the relevant one for the study of Gamma comparison, and is the key to our new De Bruijn identity for the gamma target under minimal assumptions. It also opens the way for our two main applications described in Sections 5 and 6.

4

As a first application, we obtain a new proof of the logarithmic Sobolev inequality in the gamma case for $\alpha \geq 1/2$ (see Proposition 18). Thanks to the Bakry-Emery criterion, this functional inequality is known to hold for $\alpha \geq 1/2$ with a constant independent of α (see [3]). The method of the proof developed in Section 5 extends the well-known Cauchy-Schwarz argument of the Gaussian case. Namely, we obtain a new representation for the Laguerre semigroup thanks to the following identity in law (see Proposition 6):

$$X_\tau \overset{\mathcal{L}}{=} (1-\tau)\gamma\left(\alpha - \frac{1}{2}, \lambda\right) + \left(\sqrt{\tau}\sqrt{X} + \sqrt{\frac{1-\tau}{2\lambda}}Z\right)^2.$$

This equality allows us to derive an appropriate intertwining relation which leads to a fundamental sub-commutation inequality in order to control the standardized Fisher information structure along the gamma smart path by the standardized Fisher information structure of X (see Lemma 17 and Proposition 18).

The second and final application presented in this paper uses the previous stochastic representation for the gamma smart path, we are able to derive a new HSI inequality for the gamma law with $\alpha \geq 1/2$ (see Theorem 23). HSI inequalities for different types of probability law were introduced for the first time in [25]. They allow to link the relative entropy (H) and the standardized Fisher information (I) to another kind of distance between probability measures, namely the Stein discrepancy (S). This Stein discrepancy is defined by means of a natural implicit quantity from Stein approximation method, the Stein kernel (see [13, 16, 29, 17, 18, 31, 32, 33]). Moreover, in the Gaussian case, it is proved in [25] that this inequality improves upon the classical logarithmic Sobolev inequality (see Theorem 2.2). Regarding the gamma case, the authors of [25] obtain an HSI inequality in Proposition 4.3. Our result is different from theirs since we do not use the same Stein kernel. Namely, for random variables with values in $(0, +\infty)$, ours is defined via the integration by parts identity:

$$\forall \phi \in \mathcal{C}_c^\infty\left((0, +\infty)\right), \ \mathbb{E}[(\lambda X - \alpha + \frac{1}{2})\phi(X)] = \mathbb{E}[\tau_X(X)\sqrt{X}(\partial_x^\sigma)^*(\phi)(X)],$$

where $(\partial_x^\sigma)^*(.) = \frac{d}{dx}(\sqrt{x}.)$. Let us comment briefly on this choice. First of all, we note that the Laguerre operator can be rewritten using the operator $\partial_x^\sigma = \sqrt{x}\frac{d}{dx}$ in the following way:

$$\mathcal{L}_{\alpha,\lambda}(\phi)(u) = (\partial_u^\sigma)^2(\phi)(u) + (\alpha - \frac{1}{2} - \lambda u)\partial_u(\phi)(u).$$

Moreover, plugging ϕ' in the definition of the Stein kernel, we have:

$$\mathbb{E}[(\lambda X - \alpha + \frac{1}{2})\phi'(X)] = \mathbb{E}[\tau_X(X)(\partial_x^\sigma)^2(\phi)(X)].$$

In particular, for a gamma random variable with parameters (α, λ), we have:

$$\mathbb{E}[(\lambda \gamma_{\alpha,\lambda} - \alpha + \frac{1}{2})\phi'(\gamma_{\alpha,\lambda})] = \mathbb{E}[(\partial_x^\sigma)^2(\phi)(\gamma_{\alpha,\lambda})].$$

Thus, the proximity of $\tau_X(X)$ to 1 should indicate the proximity (in law) of X to the random variable $\gamma_{\alpha,\lambda}$. The Stein discrepancy is then defined as the quadratic distance between $\tau_X(X)$ and 1. The HSI inequality obtained in Theorem 23 realizes this intuition by providing an explicit bound of the relative entropy of X with respect to $\gamma_{\alpha,\lambda}$ in terms of the Stein discrepancy and the standardized Fisher information of X. It improves upon the logarithmic Sobolev inequality for the gamma case as obtained in Proposition 18. Moreover, the appearance of the differential operator ∂_x^σ is canonically linked to the Laguerre dynamic. Indeed, under the action of this operator, the Laguerre semigroup admits stochastic representations such as the intertwining relation of Lemma 17 and the Bismut-type formulae of Lemma 20. These representations are pivotal to establish a fundamental representation of the standardized Fisher information structure along the smart path (see Proposition 22).

1.2 Structure of the paper

The article is organized as follows. In the next section, we define the stochastic representation of the interpolation scheme along the Laguerre dynamic and derive several identities in law as well as standard properties. Section 3 is devoted to the study of the Fisher information structure along the gamma smart path. In Section 4, we prove the local and integrated version of De Bruijn formulae for the gamma case. Section 5 contains the new proof of the logarithmic Sobolev inequality for the gamma case with $\alpha \geq 1/2$ and Section 6 contains the tools in order to establish the new HSI inequality for the gamma law with $\alpha \geq 1/2$. Finally, Section 7 collects the more technical proofs regarding analytical properties of the density of X_τ.

2 Gamma Interpolation

In this section, we consider a probability distribution on $]0, +\infty[$ denoted by $\mathbb{P}_X$. We denote by X the associated random variable. In the next definition, we introduce the parametrized random variable, namely the gamma smart path, which "interpolates" between the gamma law and the random variable X. This random variable X_τ, with $\tau \in (0, 1)$, is built via a three-stage explicit procedure depending on X.

Definition 1 (The gamma smart path). *Let $\tau \in (0, 1)$. Let $x \in (0, +\infty)$ be drawn according to the law of X. Let $K(\tau, x, \lambda)$ be a Poisson random variable of parameter $x\lambda\tau/(1-\tau)$ independent of X. Let $Y(\tau, x, \lambda)$ be a random variable which is drawn in the following way:*

$$Y(\tau, x, \lambda) = \begin{cases} 0 & K(\tau, x, \lambda) = 0 \\ \tilde{\gamma}\left(k, \frac{\lambda\tau}{1-\tau}\right) & K(\tau, x, \lambda) = k, \end{cases}$$

where $\tilde{\gamma}\left(k, \frac{\lambda\tau}{1-\tau}\right)$ is a gamma random variable independent of $\{X, K(\tau, x, \lambda)\}$. Then, we define X_τ by:

$$X_\tau = (1-\tau)\gamma(\alpha, \lambda) + \tau Y(\tau, X, \lambda),$$

where $\gamma(\alpha, \lambda)$ is a gamma random variable independent of $\{X, K(\tau, x, \lambda), \tilde{\gamma}\left(K(\tau, x, \lambda), \frac{\lambda\tau}{1-\tau}\right)\}$. Moreover, we denote the density of X_τ by $g^{(\alpha, \lambda)}(\tau, .)$ which is completely characterized by the following formulae:

$$\forall u > 0, \ g^{(\alpha, \lambda)}(\tau, u) = \frac{\lambda}{1-\tau}\left(\frac{u}{\tau}\right)^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda u}{1-\tau}\right)$$

$$\times \int_0^{+\infty} \left(\frac{1}{x}\right)^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right) \mathbb{P}_X(dx), \tag{5}$$

$$\forall \mu > 0, \ L(g^{(\alpha, \lambda)}(\tau, .))(\mu) = \frac{1}{\left(1 + \frac{\mu}{\lambda}(1-\tau)\right)^\alpha} \int_0^{+\infty} \exp\left(-\frac{\mu\tau x}{1 + \frac{\mu}{\lambda}(1-\tau)}\right) \mathbb{P}_X(dx), \tag{6}$$

with L being the Laplace transform operator.

Remark 1. *From the definition of the random variable X_τ, it is easy to compute its Laplace transform providing formula (6). Moreover, using the series expansion of the modified Bessel function of the first kind of order $\alpha - 1$, it is easy to compute the Laplace transform of the right-hand side of formula (5) leading to the corresponding formula for the density of X_τ. The details are left to the interested reader.*

In the next simple lemma, we prove that the law of the random variable X_τ interpolates between a gamma law of parameters (α, λ) and the probability measure $\mathbb{P}_X$.

Lemma 2. *We have:*

$$\lim_{\tau \to 0^+} X_\tau \stackrel{\mathcal{L}}{=} \gamma(\alpha, \lambda),$$

$$\lim_{\tau \to 1^-} X_\tau \stackrel{\mathcal{L}}{=} X.$$

Proof. Let $\mu > 0$. The Laplace transform of $g^{(\alpha,\lambda)}(\tau, .)$ is given by (6):

$$\forall \mu > 0, \ L(g^{(\alpha,\lambda)}(\tau, .))(\mu) = \frac{1}{\left(1 + \frac{\mu}{\lambda}(1 - \tau)\right)^\alpha} \int_0^{+\infty} \exp\left(-\frac{\mu \tau x}{1 + \frac{\mu}{\lambda}(1 - \tau)}\right) \mathbb{P}_X(dx).$$

Note that, for every $x \neq 0$:

$$\lim_{\tau \to 0^+} \frac{1}{\left(1 + \frac{\mu}{\lambda}(1 - \tau)\right)^\alpha} \exp\left(-\frac{\mu \tau x}{1 + \frac{\mu}{\lambda}(1 - \tau)}\right) = \frac{1}{\left(1 + \frac{\mu}{\lambda}\right)^\alpha} = L(\gamma(\alpha, \lambda))(\mu),$$

$$\lim_{\tau \to 1^-} \frac{1}{\left(1 + \frac{\mu}{\lambda}(1 - \tau)\right)^\alpha} \exp\left(-\frac{\mu \tau x}{1 + \frac{\mu}{\lambda}(1 - \tau)}\right) = \exp(-\mu x),$$

$$\left| \frac{1}{\left(1 + \frac{\mu}{\lambda}(1 - \tau)\right)^\alpha} \exp\left(-\frac{\mu \tau x}{1 + \frac{\mu}{\lambda}(1 - \tau)}\right) \right| \leq 1.$$

Thus, by Lebesgue dominated convergence theorem, we obtain:

$$\lim_{\tau \to 0^+} L(g^{(\alpha,\lambda)}(\tau, .))(\mu) = L(\gamma(\alpha, \lambda))(\mu),$$

$$\lim_{\tau \to 1^-} L(g^{(\alpha,\lambda)}(\tau, .))(\mu) = L(\mathbb{P}_X)(\mu).$$

We conclude by a Laplace transform version of Levy Theorem for probability measure on $(0, +\infty)$. $\qquad\square$

As a direct application of Definition 1, we obtain the following formula for the mean of the smart path X_τ.

Corollary 3. *For any $\tau > 0$, we have:*

$$\mathbb{E}\big[X_\tau\big] = (1 - \tau)\frac{\alpha}{\lambda} + \tau \mathbb{E}\big[X\big].$$

We note as well the following nice property regarding convolutions.

Corollary 4. *Let $\{X_i, \ i \in 1, ..., N\}$ be a collection of independent random variables almost surely positive. Let $X = \sum_{i=1}^N X_i$. Then, we have:*

$$X_\tau(X) \stackrel{\mathcal{L}}{=} (1 - \tau)\gamma(\alpha, \lambda) + \tau \sum_{i=1}^N Y^i(\tau, X_i, \lambda),$$

$$\stackrel{\mathcal{L}}{=} \sum_{i=1}^N \left((1 - \tau)\gamma^i\big(\frac{\alpha}{N}, \lambda\big) + \tau Y^i(\tau, X_i, \lambda)\right).$$

Proof. This is a direct application of the Laplace transform formula for X_τ, property of Laplace transform on convolutions and the definition of $Y^i(\tau, X_i, \lambda)$. $\qquad\square$

In the particular cases when $\alpha = p/2$, we obtain another representation in law for the smart path, X_τ which could be of interest. This is linked with the classical fact that the squared radial Ornstein-Uhlenbeck process (or Laguerre process) of parameters $(p/2, \lambda)$ can be represented as the squared of the euclidean norm of a p-dimensional Ornstein-Uhlenbeck process with parameter λ. Indeed, when the law of the random variable X admits a density, f_X, the density of X_τ is exactly the Lebesgue adjoint of the Laguerre semigroup acting on f_X after the time change $t = -\log(\tau)/\lambda$.

Proposition 5. *Let $\alpha = p/2$ with $p \in \mathbb{N}^*$ and $\lambda > 0$. Let $(Z_1, ..., Z_p)$ be a Gaussian random vector with mean zero and the identity matrix as its covariance matrix. Then, we have:*

$$X_\tau \overset{\mathcal{L}}{=} \sum_{i=1}^{p} \left(\sqrt{\tau} \sqrt{\frac{X}{p}} + \sqrt{\frac{1-\tau}{2\lambda}} Z_i \right)^2 .$$

Proof. Let us prove that the right hand side of the previous equality as the same Laplace transform as X_τ. By definition, we have:

$$\forall \mu > 0, \; \mathbb{E}\left[\exp\left(-\mu \sum_{i=1}^{p} \left(\sqrt{\tau} \sqrt{\frac{X}{p}} + \sqrt{\frac{1-\tau}{2\lambda}} Z_i \right)^2 \right) \right]$$

$$= \int_0^\infty \mathbb{E}\left[\exp\left(-\mu \sum_{i=1}^{p} \left(\sqrt{\tau} \sqrt{\frac{x}{p}} + \sqrt{\frac{1-\tau}{2\lambda}} Z_i \right)^2 \right) \right] \mathbb{P}_X(dx),$$

$$= \int_0^\infty \mathbb{E}\left[\exp\left(-\mu \frac{(1-\tau)}{2\lambda} \sum_{i=1}^{p} \frac{\left(\sqrt{\tau} \sqrt{\frac{x}{p}} + \sqrt{\frac{1-\tau}{2\lambda}} Z_i \right)^2}{\frac{1-\tau}{2\lambda}} \right) \right] \mathbb{P}_X(dx).$$

Conditionally to X, we recognize the Laplace transform of a non-central chi-squared random variable with parameters $(p, 2\lambda\tau x/(1-\tau))$ evaluated at $\mu(1-\tau)/2\lambda$. Thus, we obtain:

$$\mathbb{E}\left[\exp\left(-\mu \sum_{i=1}^{p} \left(\sqrt{\tau} \sqrt{\frac{X}{p}} + \sqrt{\frac{1-\tau}{2\lambda}} Z_i \right)^2 \right) \right] = \int_0^\infty \left(\frac{1}{1 + \frac{\mu(1-\tau)}{\lambda}} \right)^{\frac{p}{2}} \exp\left(\frac{-\mu\tau x}{1 + \frac{\mu(1-\tau)}{\lambda}} \right) \mathbb{P}_X(dx),$$

$$= \left(\frac{1}{1 + \frac{\mu(1-\tau)}{\lambda}} \right)^{\frac{p}{2}} \int_0^{+\infty} \exp\left(\frac{-\mu\tau x}{1 + \frac{\mu(1-\tau)}{\lambda}} \right) \mathbb{P}_X(dx),$$

$$= L(X_\tau)(\mu).$$

$\square$

The next result is a combination of the two previous representations which is available for $\alpha > 1/2$ et which is of central importance for Sections 5 and 6.

Corollary 6. *Let $\alpha > 1/2$ and $\lambda > 0$. Let Z be a standard normal random variable and $\gamma(\alpha - 1/2, \lambda)$ a gamma random variable of parameters $(\alpha - 1/2, \lambda)$ such that $\big(Z, X, \gamma(\alpha - 1/2, \lambda)\big)$ are independent. Then,*

$$X_\tau \overset{\mathcal{L}}{=} (1-\tau)\gamma\left(\alpha - \frac{1}{2}, \lambda\right) + \left(\sqrt{\tau}\sqrt{X} + \sqrt{\frac{1-\tau}{2\lambda}} Z \right)^2 .$$

Proof. The Laplace transform of X_τ is equal to:

$$\forall \mu > 0, L(X_\tau)(\mu) = \frac{1}{\big(1 + \frac{\mu}{\lambda}(1-\tau)\big)^\alpha} L(X)\big(\frac{\mu\tau}{1 + \frac{\mu}{\lambda}(1-\tau)}\big),$$

$$= \frac{1}{\big(1 + \frac{\mu}{\lambda}(1-\tau)\big)^{\alpha - \frac{1}{2}}} \frac{1}{\big(1 + \frac{\mu}{\lambda}(1-\tau)\big)^{\frac{1}{2}}} L(X)\big(\frac{\mu\tau}{1 + \frac{\mu}{\lambda}(1-\tau)}\big),$$

$$= L\left((1-\tau)\gamma\left(\alpha - \frac{1}{2}, \lambda\right) \right)(\mu) L(X_\tau')(\mu),$$

8

where X'_τ is a smart path of parameters $(1/2, \lambda)$ independent of $\gamma(\alpha - 1/2, \lambda)$. We apply Proposition 5 to X'_τ to conclude the proof. $\qquad\square$

We end this section by two lemmata which provide explicit formulae for the β-moments and the exponential moments of the smart path X_τ under the assumptions that such moments exist for the initial law, $\mathbb{P}_X$.

Lemma 7. *Let $\alpha > 0$, $\lambda > 0$ and β such that $\alpha + \beta > 0$. Let X be a strictly positive random variable such that $\mathbb{E}[X^\beta] < +\infty$. Then, $\mathbb{E}[(X_\tau)^\beta] < +\infty$ and we have:*

$$\mathbb{E}[(X_\tau)^\beta] = \left(\frac{\lambda}{1-\tau}\right)^{-\beta} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \int_0^{+\infty} \exp\left(-\frac{\lambda x \tau}{1-\tau}\right) {}_1F_1\left(\alpha+\beta, \alpha, \frac{\lambda \tau x}{1-\tau}\right) \mathbb{P}_X(dx).$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function (of the first kind).

Proof. By definition, we have:

$$\begin{aligned}
\mathbb{E}[(X_\tau)^\beta] &= \int_0^{+\infty} u^\beta g^{(\alpha,\lambda)}(\tau, u) du, \\
&= \frac{\lambda}{1-\tau} \left(\frac{1}{\tau}\right)^{\frac{\alpha-1}{2}} \int_0^{+\infty} \left(\frac{1}{x}\right)^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda \tau x}{1-\tau}\right) \\
&\quad \times \left(\int_0^{+\infty} u^{\beta+\frac{\alpha-1}{2}} \exp\left(-\frac{u\lambda}{1-\tau}\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{u\tau x}}{1-\tau}\right) du\right) \mathbb{P}_X(dx).
\end{aligned}$$

Expanding the modified Bessel functions of the first kind into power series, we have the following (since $\alpha + \beta > 0$):

$$\begin{aligned}
&\int_0^{+\infty} u^{\beta+\frac{\alpha-1}{2}} \exp\left(-\frac{u\lambda}{1-\tau}\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{u\tau x}}{1-\tau}\right) du \\
&= \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{1}{\Gamma(\alpha+n)} \left(\frac{\lambda\sqrt{x\tau}}{1-\tau}\right)^{\alpha-1+2n} \int_0^{+\infty} u^{\beta+\alpha-1+n} \exp\left(-\frac{u\lambda}{1-\tau}\right) du, \\
&= \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{1}{\Gamma(\alpha+n)} \left(\frac{\lambda\sqrt{x\tau}}{1-\tau}\right)^{\alpha-1+2n} \left(\frac{1-\tau}{\lambda}\right)^{\alpha+\beta} \Gamma(\alpha+\beta+n), \\
&= \left(\frac{\lambda}{1-\tau}\right)^{-(\beta+1)} (x\tau)^{\frac{\alpha-1}{2}} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} {}_1F_1\left(\alpha+\beta, \alpha, \frac{\lambda\tau x}{1-\tau}\right).
\end{aligned}$$

Thus,

$$\mathbb{E}[(X_\tau)^\beta] = \left(\frac{\lambda}{1-\tau}\right)^{-\beta} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \int_0^{+\infty} \exp\left(-\frac{\lambda x \tau}{1-\tau}\right) {}_1F_1\left(\alpha+\beta, \alpha, \frac{\lambda \tau x}{1-\tau}\right) \mathbb{P}_X(dx).$$

To conclude, we need to study the finiteness of the previous integral and in particular the integrability of the integrand at $+\infty$. But, we have the following asymptotic:

$$ {}_1F_1(\alpha+\beta, \alpha, z) \underset{z\to+\infty}{\sim} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \exp(z) z^\beta.$$

Moreover, by assumptions, $\mathbb{E}[(X)^\beta] < +\infty$. This concludes the proof of the lemma. $\qquad\square$

Lemma 8. *Let $\mu > 0$ such that $\mathbb{E}[\exp(\mu X)] < +\infty$. Then, there exists $s(\mu, \tau) > 0$, such that:*

$$\forall \tau \in (0,1), \ \mathbb{E}[e^{s(\mu,\tau)X_\tau}] < +\infty.$$

9

Moroever, we have:

$$\forall \tau \in (0,1), \ \mathbb{E}\big[e^{s(\mu,\tau)X_\tau}\big] = \frac{1}{\left(1 + \frac{\mu(1-\tau)}{\lambda\tau}\right)^\alpha} \mathbb{E}[e^{\mu X}],$$

$$s(\mu,\tau) = \frac{\mu}{\tau + \frac{\mu(1-\tau)}{\lambda}}.$$

Proof. Let $s(\mu,\tau)$ be as in the lemma. By definition, we have:

$$\forall \tau \in (0,1), \ \mathbb{E}\big[e^{s(\mu,\tau)X_\tau}\big] = \left(\frac{\lambda}{1-\tau}\right)\left(\frac{1}{\tau}\right)^{\frac{\alpha-1}{2}} \int_0^{+\infty} \left(\frac{1}{x}\right)^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau x}{1-\tau}\right)$$
$$\times \left(\int_0^{+\infty} u^{\frac{\alpha-1}{2}} \exp\left(-u\big(\frac{\lambda}{1-\tau} - s(\mu,\tau)\big)\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right) du\right) \mathbb{P}_X(dx).$$

Expanding the modified Bessel function of the first kind of order $\alpha - 1$ into power series, we obtain:

$$\int_0^{+\infty} u^{\frac{\alpha-1}{2}} \exp\left(-u\big(\frac{\lambda}{1-\tau} - s(\mu,\tau)\big)\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right) du = \sum_{n=0}^{+\infty} \frac{1}{n!}\frac{1}{\Gamma(\alpha+n)}\left(\frac{\lambda\sqrt{\tau x}}{1-\tau}\right)^{\alpha-1+2n}$$
$$\times \int_0^{+\infty} u^{\alpha-1+n} \exp\left(-u\big(\frac{\lambda}{1-\tau} - s(\mu,\tau)\big)\right) du.$$

Note that the integral on the right hand side is finite since $s(\mu,\tau) < \lambda/(1-\tau)$. Thus,

$$\int_0^{+\infty} u^{\frac{\alpha-1}{2}} \exp\left(-u\big(\frac{\lambda}{1-\tau} - s(\mu,\tau)\big)\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right) du = \left(\frac{\lambda\sqrt{x\tau}}{1-\tau}\right)^{\alpha-1} \frac{1}{\left(\frac{\lambda}{1-\tau} - s(\mu,\tau)\right)^\alpha}$$
$$\times \exp\left(\frac{\lambda\tau x}{1-\tau} \frac{1}{1 - \frac{s(\mu,\tau)(1-\tau)}{\lambda}}\right).$$

Consequently, we have:

$$\mathbb{E}\big[e^{s(\mu,\tau)X_\tau}\big] = \frac{1}{\left(1 - \frac{s(\mu,\tau)(1-\tau)}{\lambda}\right)^\alpha} \int_0^{+\infty} \exp\left(\frac{s(\mu,\tau)x\tau}{1 - \frac{s(\mu,\tau)(1-\tau)}{\lambda}}\right) \mathbb{P}_X(dx).$$

Using the formula for $s(\mu,\tau)$, we obtain the desired result. $\qquad\square$

3 Fisher Information structure along the gamma smart path

In this subsection, we introduce a localized version of Fisher information which is relevant in order to establish the De Bruijn formula for the gamma law. Indeed, this quantity appears naturally at a local level along the smart path, X_τ, when computing the derivative of the relative entropy $D(X_\tau \| \gamma_{\alpha,\lambda})$ with respect to τ.

Definition 2. *Let $\tau \in (0,1)$ and X be a positive random variable with density f_X. We define $I_\gamma^\tau(X)$ by:*

$$I_\gamma^\tau(X) = \mathbb{E}\big[X\big(\rho_X(X) - \rho_{\alpha,\frac{\lambda}{(1-\tau)}}(X)\big)^2\big],$$

where $\rho_X(u) = \partial_u\big(\log(f_X(u))\big)$ and $\rho_{\alpha,\lambda/(1-\tau)}(u) = (\alpha-1)/u - \lambda/(1-\tau)$.

Remark 9. • *Note that if X has a gamma law with parameters (α, λ), we have:*

$$I_\gamma^\tau(X) = \mathbb{E}\Big[X\big(\frac{\alpha-1}{X} - \lambda - \frac{\alpha-1}{X} + \frac{\lambda}{1-\tau}\big)^2\Big],$$

$$= \frac{\lambda^2\tau^2}{(1-\tau)^2}\mathbb{E}[X],$$

$$= \lambda\alpha\frac{\tau^2}{(1-\tau)^2}.$$

• *Note that the previous quantity is actually well-defined for any values of α whereas the quantities $\mathbb{E}[X(\rho_X(X))^2]$ and $\mathbb{E}[(\rho_X(X))^2]$ are finite only for gamma laws with shape parameters $\alpha > 1$ and $\alpha > 2$ respectively.*

Thus, we introduce the following standardized localized Fisher information with respect to the gamma law of parameters (α, λ):

$$J_{st,\gamma}(X) = \mathbb{E}\Big[X\big(\rho_X(X) - \rho_{\alpha,\frac{\lambda}{(1-\tau)}}(X) - \frac{\lambda\tau}{1-\tau}\big)^2\Big].$$

We note that this standardized localized Fisher information is actually equal to:

$$J_{st,\gamma}(X) = \mathbb{E}\big[X\big(\rho_X(X) - \rho_{\alpha,\lambda}(X)\big)^2\big].$$

Regarding these quantities along the smart path, we have the following results.

Proposition 10. *Assume that X admits a first moment. Then, we have,*

$$\forall\tau \in (0,1),\ I_\gamma^\tau(X_\tau) < +\infty$$

Moreover, if $\mathbb{E}[X] = \alpha/\lambda$, we have:

$$\forall\alpha \geq 1,\ I_\gamma^\tau(X_\tau) \leq \frac{\alpha\lambda\tau}{(1-\tau)^2},$$

$$\forall\alpha \in (0,1),\ I_\gamma^\tau(X_\tau) \leq \Big(1 + \frac{1}{\alpha}\Big)\frac{\alpha\lambda\tau}{(1-\tau)^2}.$$

Proof. By definition of $I_\gamma^\tau(.)$, we have:

$$I_\gamma^\tau(X_\tau) = \mathbb{E}\big[X_\tau\big(\rho_{X_\tau}(X_\tau) - \rho_{\alpha,\frac{\lambda}{(1-\tau)}}(X_\tau)\big)^2\big],$$

$$= \mathbb{E}\big[\frac{1}{X_\tau}\big(\overline{\rho}_{X_\tau}(X_\tau) - \overline{\rho}_{\alpha,\frac{\lambda}{(1-\tau)}}(X_\tau)\big)^2\big],$$

where $\overline{\rho}_X(u) = \partial_u\big(uf_X(u)\big)/f_X(u)$ and $\overline{\rho}_{\alpha,\frac{\lambda}{(1-\tau)}}(u) = \alpha - \lambda u/(1-\tau)$. Note that:

$$\overline{\rho}_X(u) = u\rho_X(u) + 1,$$
$$\forall\phi \in C_c^\infty\big((0,+\infty)\big),\ \mathbb{E}\big[\overline{\rho}_X(X)\phi(X)\big] = -\mathbb{E}\big[X\phi^{(1)}(X)\big],$$
$$\forall\phi \in C_c^\infty\big((0,+\infty)\big),\ \mathbb{E}\big[(\alpha - \lambda\gamma(\alpha,\lambda))\phi(\gamma(\alpha,\lambda))\big] = -\mathbb{E}\big[\gamma(\alpha,\lambda)\phi^{(1)}(\gamma(\alpha,\lambda))\big].$$

Let $\phi \in C_c^\infty\big((0,+\infty)\big)$. We have:

$$\mathbb{E}\big[\overline{\rho}_{X_\tau}(X_\tau)\phi(X_\tau)\big] = -\mathbb{E}\big[X_\tau\phi^{(1)}(X_\tau)\big],$$

$$= -(1-\tau)\mathbb{E}\big[\gamma(\alpha,\lambda)\phi^{(1)}(X_\tau)\big] - \tau\mathbb{E}\big[Y(\tau,X,\lambda)\phi^{(1)}(X_\tau)\big],$$

$$= \mathbb{E}\big[(\alpha - \lambda\gamma(\alpha,\lambda))\phi(X_\tau)\big] - \tau\mathbb{E}\big[Y(\tau,X,\lambda)\phi^{(1)}(X_\tau)\big],$$

where we have used Definition 1 and Stein formula for the Gamma distribution. Let us deal with the second term in details:

$$\mathbb{E}\big[Y(\tau, X, \lambda)\phi^{(1)}((1-\tau)\gamma(\alpha, \lambda) + \tau Y(\tau, X, \lambda))\big]$$

$$= \int_0^{+\infty} \mathbb{E}\big[Y(\tau, x, \lambda)\phi^{(1)}((1-\tau)\gamma(\alpha, \lambda) + \tau Y(\tau, x, \lambda))\big]\mathbb{P}_X(dx),$$

$$= \sum_{k=1}^{+\infty} \int_0^{+\infty} \mathbb{E}\big[\gamma(k, \frac{\lambda\tau}{1-\tau})\phi^{(1)}((1-\tau)\gamma(\alpha, \lambda) + \tau\gamma(k, \frac{\lambda\tau}{1-\tau}))\big]$$
$$\times \mathbb{P}\big(K(\tau, x, \lambda) = k\big)\mathbb{P}_X(dx),$$

$$= \frac{1}{\tau}\sum_{k=1}^{+\infty} \int_0^{+\infty} \mathbb{E}\big[(\frac{\lambda\tau}{1-\tau}\gamma(k, \frac{\lambda\tau}{1-\tau}) - k)\phi((1-\tau)\gamma(\alpha, \lambda)$$
$$+ \tau\gamma(k, \frac{\lambda\tau}{1-\tau}))\big] \times \mathbb{P}\big(K(\tau, x, \lambda) = k\big)\mathbb{P}_X(dx),$$

$$= \frac{1}{\tau}\mathbb{E}\big[(\frac{\lambda\tau}{1-\tau}Y(\tau, X, \lambda) - K(\tau, X, \lambda))\phi(X_\tau)\big].$$

Thus, we obtain:

$$\mathbb{E}\big[\overline{\rho}_{X_\tau}(X_\tau)\phi(X_\tau)\big] = \mathbb{E}\big[(\alpha - \lambda\gamma(\alpha, \lambda) + K(\tau, X, \lambda) - \frac{\lambda\tau}{1-\tau}Y(\tau, X, \lambda))\phi(X_\tau)\big],$$

$$= \mathbb{E}\big[(\alpha + K(\tau, X, \lambda) - \frac{\lambda}{1-\tau}X_\tau)\phi(X_\tau)\big].$$

Therefore, we have:

$$I_\gamma^\tau(X_\tau) = \mathbb{E}\big[\frac{1}{X_\tau}\big(\mathbb{E}\big[K(\tau, X, \lambda)|X_\tau\big]\big)^2\big],$$

$$\leq \mathbb{E}\big[\frac{K(\tau, X, \lambda)^2}{X_\tau}\big],$$

$$\leq \sum_{k=1}^{+\infty} \int_0^{+\infty} \mathbb{P}\big(K(\tau, x, \lambda) = k\big)\mathbb{E}\big[\frac{k^2}{\gamma(\alpha + k, \frac{\lambda}{1-\tau})}\big]\mathbb{P}_X(dx),$$

$$\leq \frac{\lambda}{1-\tau}\sum_{k=1}^{+\infty} \int_0^{+\infty} \Big(\frac{\lambda x\tau}{1-\tau}\Big)^k \frac{1}{k!}\exp\Big(-\frac{\lambda x\tau}{1-\tau}\Big)\frac{k^2}{\alpha + k - 1}\mathbb{P}_X(dx),$$

$$\leq \frac{\lambda^2\tau}{(1-\tau)^2} \int_0^{+\infty} \Big(\sum_{k=0}^{+\infty}\Big(\frac{\lambda x\tau}{1-\tau}\Big)^k\frac{1}{k!}\frac{k+1}{k+\alpha}\Big)\exp\Big(-\frac{\lambda x\tau}{1-\tau}\Big)x\mathbb{P}_X(dx).$$

If $\alpha \geq 1$, we have the following bound:

$$I_\gamma^\tau(X_\tau) \leq \frac{\lambda^2\tau}{(1-\tau)^2}\mathbb{E}\big[X\big] < +\infty.$$

If $\alpha \in (0, 1)$, we have:

$$I_\gamma^\tau(X_\tau) \leq \frac{\lambda^2\tau}{(1-\tau)^2}\Big(1 + \frac{1}{\alpha}\Big)\mathbb{E}\big[X\big] < +\infty$$

This ends the proof of the proposition. $\square$

Proposition 11. *Assume that X admits a first moment and that $\mathbb{E}\big[X\big] = \alpha/\lambda$. Then, we have:*

$$J_{st,\gamma}(X_\tau) = I_\gamma^\tau(X_\tau) - \frac{\alpha\lambda\tau^2}{(1-\tau)^2} \geq 0.$$

Moreover, when $\alpha \geq 1$, we have the following upper bound:

$$J_{st,\gamma}(X_\tau) \leq \frac{\alpha\lambda\tau}{1-\tau}.$$

Proof. By definition of the standardized localized Fisher information, we have:

$$J_{st,\gamma}(X_\tau) = \mathbb{E}\big[X_\tau\big(\rho_{X_\tau}(X_\tau) - \rho_{\alpha,\frac{\lambda}{(1-\tau)}}(X_\tau) - \frac{\lambda\tau}{1-\tau}\big)^2\big],$$

$$= I_\gamma^\tau(X_\tau) - 2\frac{\lambda\tau}{1-\tau}\mathbb{E}\big[X_\tau\big(\rho_{X_\tau}(X_\tau) - \rho_{\alpha,\frac{\lambda}{(1-\tau)}}(X_\tau)\big)\big] + \mathbb{E}\big[X_\tau\big]\frac{\lambda^2\tau^2}{(1-\tau)^2},$$

$$= I_\gamma^\tau(X_\tau) - 2\frac{\lambda\tau}{1-\tau}\Big(-\alpha + \frac{\lambda}{1-\tau}\mathbb{E}\big[X_\tau\big]\Big) + \frac{\alpha\lambda\tau^2}{(1-\tau)^2},$$

$$= I_\gamma^\tau(X_\tau) - \frac{\alpha\lambda\tau^2}{(1-\tau)^2}.$$

where we have used Corollary 3 as well as a classical property of the score function. $\qquad\square$

Remark 12. : *The previous result actually holds for any positive random variable with finite first moment equal to α/λ and with finite localized Fisher information. Namely, we have:*

$$J_{st,\gamma}(X) = I_\gamma^\tau(X) - \frac{\alpha\lambda\tau^2}{(1-\tau)^2}.$$

4   A new formulation of De Bruijn identity

By definition, for every $\tau \in (a, b)$, we have:

$$D(X_\tau\|\gamma_{\alpha,\lambda}) = \int_0^{+\infty} g^{(\alpha,\lambda)}(\tau, u)\log\left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)}\right)du.$$

Note that $D(X_\tau\|\gamma_{\alpha,\lambda})$ is finite by Lemma 27, Lemma 28 and together with the assumption that X has finite $\alpha + 4$ moment (which ensures that X_τ has finite $\alpha + 4$ moment by Lemma 7). In order to obtain a De Bruijn formula we need to be able to interchange derivatives and integrals in the above and thus bound the integrand uniformly in τ. In doing so we will identify a moment condition on X which is the only assumption that will be necessary for our formula to hold.

Theorem 13. *Let X be an almost surely positive random variable with finite $\alpha + 4$ moment. Then, we have:*

$$\frac{d}{d\tau}\Big(D(X_\tau\|\gamma_{\alpha,\lambda})\Big) = \frac{1}{\lambda\tau}J_{st,\gamma}(X_\tau).$$

Remark 14. *As already discussed in the Introduction, this local version of De Bruijn formula should be compared with Proposition 5.2.2 in [5] where a De Bruijn formula for the gamma law has been obtained by semigroup arguments. Although formally equivalent, our result is much more general general as it holds under moment conditions only, whether a density exists or not for $\mathbb{P}_X$. Indeed, Proposition 5.2.2 in [5] requires existence of a density which, moreover, must be in the domain of the Dirichlet form associated with the Laguerre generator.*

Proof of Theorem 13, part I : interchange of derivative and integral
We want to exchange the order between the differentiation with respect to τ and the integration with respect to u. For this purpose, we need to control uniformly in τ the quantity

$$\partial_\tau\big(g^{(\alpha,\lambda)}(\tau, u)\log\big(g^{(\alpha,\lambda)}(\tau, u)/\gamma_{\alpha,\lambda}(u)\big)\big).$$

By standard computations, we have:

$$\left| \partial_\tau \left(g^{(\alpha,\lambda)}(\tau, u) \log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) \right) \right| \le (I) + (II) + (III),$$

with,

$$(I) = |\partial_\tau \left(g^{(\alpha,\lambda)}(\tau, u) \right)| \left| \log \left(g^{(\alpha,\lambda)}(\tau, u) \right) \right|,$$
$$(II) = |\partial_\tau \left(g^{(\alpha,\lambda)}(\tau, u) \right)| \left| \log \left(\gamma_{\alpha,\lambda}(u) \right) \right|,$$
$$(III) = |\partial_\tau \left(g^{(\alpha,\lambda)}(\tau, u) \right)|.$$

Let us deal with the first term, (I). The others will follow similarly. By Proposition 25, we have:

$$|\partial_\tau \left(g^{(\alpha,\lambda)}(\tau, u) \right)| \le P_{\alpha,\lambda,a,b}(u) g^{(\alpha,\lambda)}(\tau, u) + Q_{\alpha,\lambda,a,b}(u) h^{(\alpha,\lambda)}(\tau, u) + \frac{u}{\lambda a} k^{(\alpha,\lambda)}(\tau, u),$$

with $P_{\alpha,\lambda,a,b}(.)$ and $Q_{\alpha,\lambda,a,b}(.)$ polynomials in u of degree 1 with positive coefficients. Moreover, by Lemma 28, we obtain:

$$(I) \le \left[A_{X,\alpha,\lambda,a,b} + 2\frac{\lambda u}{1-b} + |\alpha - 1| |\log(u)| \right] \left[P_{\alpha,\lambda,a,b}(u) g^{(\alpha,\lambda)}(\tau, u) + Q_{\alpha,\lambda,a,b}(u) h^{(\alpha,\lambda)}(\tau, u) + \frac{u}{\lambda a} k^{(\alpha,\lambda)}(\tau, u) \right].$$

Thus, if we can prove that $u^2 k^{(\alpha,\lambda)}(\tau, u)$ is dominated uniformly in τ by an integrable function on $(0, +\infty)$, the first step will be done. By Lemma 27, we have:

$$u^2 k^{(\alpha,\lambda)}(\tau, u) \le C^3_{a,b,\lambda,\alpha} u^2 \left(u^{\alpha+1} \exp\left(-\lambda \frac{u}{4(1-a)} \right) + u^{\alpha-1} \mathbb{E}[X^2 \mathbb{I}_{\{X > \frac{u}{4}\}}] \right).$$

The first term is clearly in $L^1 \left((0, +\infty) \right)$. Moreover, for the second term, we have:

$$\int_0^{+\infty} u^{\alpha+1} \mathbb{E}[X^2 \mathbb{I}_{\{X > \frac{u}{4}\}}] du = \mathbb{E}\left[X^2 \int_0^{+\infty} u^{\alpha+1} \mathbb{I}_{\{X > \frac{u}{4}\}} du \right],$$
$$= \mathbb{E}\left[X^2 \int_0^{4X} u^{\alpha+1} du \right],$$
$$= \frac{4^{\alpha+2}}{\alpha+2} \mathbb{E}[X^{\alpha+4}] < +\infty.$$

Proof of Theorem 13, part II : Integration by parts
We proved that one can interchange derivatives and integrals in the expression for relative entropy to obtain:

$$\frac{d}{d\tau} \left(D(X_\tau \| \gamma_{\alpha,\lambda}) \right) = \int_0^{+\infty} \partial_\tau \left(g^{(\alpha,\lambda)}(\tau, u) \log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) \right) du,$$
$$= \int_0^{+\infty} \partial_\tau \left(g^{(\alpha,\lambda)}(\tau, u) \right) \log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) du,$$
$$= -\frac{1}{\lambda \tau} \int_0^{+\infty} \partial_u \left(g^{(\alpha,\lambda)}(\tau, u) u \partial_u \left(\log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) \right) \right) \log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) du,$$

where we have used Proposition 25 in the last equality. Now, we perform cautiously an integration by parts. Let $R > 1$ be big enough. For any $u \in [1/R, R]$, we have:

$$\partial_u \left(g^{(\alpha,\lambda)}(\tau, u) u \partial_u \left(\log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) \right) \log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) \right)$$
$$= \partial_u \left(g^{(\alpha,\lambda)}(\tau, u) u \partial_u \left(\log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) \right) \right)$$
$$\times \log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) + g^{(\alpha,\lambda)}(\tau, u) u \left[\partial_u \left(\log \left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)} \right) \right) \right]^2.$$

By the previous step, the first term is clearly integrable on $(0, +\infty)$. Let us compute explicitly the second term and study its integrability on $(0, +\infty)$. We have:

$$g^{(\alpha,\lambda)}(\tau, u)u\left[\partial_u\left(\log\left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)}\right)\right)\right]^2 = g^{(\alpha,\lambda)}(\tau, u)u\left[\frac{\partial_u(g^{(\alpha,\lambda)})(\tau, u)}{g^{(\alpha,\lambda)}(\tau, u)} - \frac{\gamma'_{\alpha,\lambda}(u)}{\gamma_{\alpha,\lambda}(u)}\right]^2,$$

$$= g^{(\alpha,\lambda)}(\tau, u)u\left[\frac{\partial_u(g^{(\alpha,\lambda)})(\tau, u)}{g^{(\alpha,\lambda)}(\tau, u)} - \frac{\alpha-1}{u} + \lambda\right]^2.$$

Moreover, by Proposition 25, we have:

$$\partial_u(g^{(\alpha,\lambda)})(\tau, u) = \frac{\alpha-1}{u}g^{(\alpha,\lambda)}(\tau, u) - \frac{\lambda}{1-\tau}g^{(\alpha,\lambda)}(\tau, u) + h^{(\alpha,\lambda)}(\tau, u).$$

Thus,

$$g^{(\alpha,\lambda)}(\tau, u)u\left[\partial_u\left(\log\left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)}\right)\right)\right]^2$$

$$= g^{(\alpha,\lambda)}(\tau, u)u\left[\frac{h^{(\alpha,\lambda)}(\tau, u)}{g^{(\alpha,\lambda)}(\tau, u)} - \frac{\lambda\tau}{1-\tau}\right]^2$$

$$= u\frac{(h^{(\alpha,\lambda)}(\tau, u))^2}{g^{(\alpha,\lambda)}(\tau, u)} - 2uh^{(\alpha,\lambda)}(\tau, u)\frac{\lambda\tau}{1-\tau} + ug^{(\alpha,\lambda)}(\tau, u)\left(\frac{\lambda\tau}{1-\tau}\right)^2.$$

The second and the third terms are clearly integrable (see Lemma 27). For the first term, we note that:

$$\int_0^{+\infty} ug^{(\alpha,\lambda)}(\tau, u)\frac{(h^{(\alpha,\lambda)}(\tau, u))^2}{(g^{(\alpha,\lambda)}(\tau, u))^2}du = I_\gamma^\tau(X_\tau),$$

which is finite by Proposition 10.

Proof of Theorem 13, part III : conclusion

We need to study the limits at 0^+ and at $+\infty$ of the following function:

$$u \longmapsto ug^{(\alpha,\lambda)}(\tau, u)\frac{\partial}{\partial u}\left(\log\left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)}\right)\right)\log\left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)}\right).$$

The limits exist by the proof of Theorem 22. Moreover, $u \to \sqrt{ug^{(\alpha,\lambda)}(\tau, u)}\partial_u\left(\log\left(g^{(\alpha,\lambda)}(\tau, u)/\gamma_{\alpha,\lambda}(u)\right)\right)$ is a square-integrable function on $(0, +\infty)$. Thus, we need to study the limits at 0^+ and at $+\infty$ of the function:

$$u \longmapsto \sqrt{ug^{(\alpha,\lambda)}(\tau, u)}\log\left(\frac{g^{(\alpha,\lambda)}(\tau, u)}{\gamma_{\alpha,\lambda}(u)}\right).$$

Using Lemma 26, we have:

$$ug^{(\alpha,\lambda)}(\tau, u) \underset{u\to 0^+}{\sim} C_{\tau,\alpha,\lambda}u^\alpha,$$

which ensures that the limit at 0^+ of the previous function is 0. All that remains is to prove that $\lim_{u\to+\infty} u^3g^{(\alpha,\lambda)}(\tau, u) = 0$. This step is a direct application of the Lebesgue dominated convergence theorem. Indeed, by definition, we have:

$$u^3g^{(\alpha,\lambda)}(\tau, u) = \int_0^{+\infty} \frac{\lambda}{1-\tau}\left(\frac{1}{\tau}\right)^{\frac{\alpha-1}{2}}u^{\frac{\alpha-1}{2}+3}\exp\left(-\frac{\lambda}{1-\tau}u\right)$$

$$\times \left(\frac{1}{x}\right)^{\frac{\alpha-1}{2}}\exp\left(-\frac{\lambda\tau}{1-\tau}x\right)I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx).$$

The almost everywhere convergence follows by asymptotic properties of the modified Bessel function of the first kind of order $\alpha - 1$ at infinity. Moreover, denoting by (1) the integrand, we have, $\mathbb{P}_X - a.e.$:

$$(1) \leq C_{\alpha,\lambda,\tau} u^{\alpha+2} \exp\big(-\frac{\lambda}{1-\tau}u\big) \exp\big(-\frac{\lambda\tau}{1-\tau}x\big) \cosh\big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\big),$$

$$\leq C_{\alpha,\lambda,\tau} u^{\alpha+2} \exp\big(-\frac{\lambda}{1-\tau}(\sqrt{u}-\sqrt{x\tau})^2\big),$$

$$\leq C^1_{\alpha,\lambda,\tau} + C^2_{\alpha,\lambda,\tau} x^{\alpha+2},$$

which is integrable since X admits moment of order $\alpha + 4$. $\square$

As anticipated, we conclude with a proof of an integrated De Bruijn identity.

Theorem 15. *Under the assumptions of Theorem 13 : if $\mathbb{E}[X] = \alpha/\lambda$ then*

$$D(X \| \gamma_{\alpha,\lambda}) = \int_0^1 \frac{1}{\lambda\tau} J_{st,\gamma}(X_\tau) d\tau. \tag{7}$$

Proof. First of all, note that, by Propositions 10 and 11, we can integrate $1/(\lambda\tau) J_{st,\gamma}(X_\tau)$ over any compact subsets $[a,b]$ strictly contained in $[0,1]$. Thus, we get:

$$D(X_b \| \gamma_{\alpha,\lambda}) - D(X_a \| \gamma_{\alpha,\lambda}) = \int_a^b \frac{1}{\lambda\tau} J_{st,\gamma}(X_\tau) d\tau.$$

Now, we want to prove that:

$$\lim_{b \to 1^-} D(X_b \| \gamma_{\alpha,\lambda}) = D(X \| \gamma_{\alpha,\lambda}), \quad \lim_{a \to 0^+} D(X_a \| \gamma_{\alpha,\lambda}) = 0.$$

Let us study the first limit. We know that $X_b \to X$ in distribution when $b \to 1^-$ by Lemma 2. Since the relative entropy is lower semicontinuous, we have that:

$$\liminf_{b \to 1^-} D(X_b \| \gamma_{\alpha,\lambda}) \geq D(X \| \gamma_{\alpha,\lambda}).$$

Now, by Lemma 6.2.13 of [19], we have the following representation for the relative entropy:

$$D(X \| \gamma_{\alpha,\lambda}) = \sup_{\phi \in C_b(\mathbb{R}_+)} \left(\int_0^{+\infty} \phi(x) f_X(x) dx - \log\left(\int_0^{+\infty} \exp(\phi(x)) \gamma_{\alpha,\lambda}(x) dx \right) \right).$$

Following the beginning of the proof of Lemma 1.2 in [15], using the fact that the Laguerre semigroup is a contraction on every $L^p\big(\mathbb{R}_+^*, \gamma_{\alpha,\lambda}(x)dx\big)$, for $p \geq 1$, and the fact that $P_t\big(C_b(\mathbb{R}_+)\big) \subset C_b(\mathbb{R}_+)$, we obtain:

$$D(X_b \| \gamma_{\alpha,\lambda}) \leq D(X \| \gamma_{\alpha,\lambda}).$$

Thus, the first limit is proved. To finish, let us prove that $D(X_a \| \gamma_{\alpha,\lambda}) \to 0$ as $a \to 0$. Using Definition 1 along with the known fact that entropy increases with independent convolution, we deduce that

$$H(\gamma(\alpha,\lambda)) - H(X_a) \leq H(\gamma(\alpha,\lambda)) - H((1-a)\gamma(\alpha,\lambda)) = -\log(1-a)$$

where in the last equality we used the known identity $H(cY) = H(Y) + \log c$ for $c > 0$. Likewise, still using Definition 1 and the fact that the second summand in this representation is positive, we see that

$$\mathbb{E}[\log \gamma_{\alpha,\lambda}] - \mathbb{E}[\log X_a] \leq -\log(1-a).$$

Using the following relation,

$$D(X_a\|\gamma_{\alpha,\lambda}) = H(\gamma_{\alpha,\lambda}) - H(X_a) + (\alpha - 1)\int_0^{+\infty}\big(\gamma_{\alpha,\lambda}(u) - g^{(\alpha,\lambda)}(\tau,u)\big)\log(u)du.$$

we obtain,

$$0 \leq D(X_a\|\gamma_{\alpha,\lambda}) \leq -\alpha\log(1-a)$$

and the conclusion follows. $\qquad\Box$

Remark 16. *By fixing the logarithmic moment of X to be equal to that of $\gamma_{\alpha,\lambda}$, we obtain the following straightforward identity:*

$$H(\gamma_{\alpha,\lambda}) - H(X) = \int_0^1 \frac{1}{\lambda\tau}J_{st,\gamma}(X_\tau)d\tau,$$

where H is the Shannon entropy.

5   A new proof of LSI for the gamma case with $\alpha \geq \frac{1}{2}$

Before stating the main result of this section, we introduce some notations. Let $\mathcal{B}\big((0,+\infty)\big)$ be the set of bounded measurable functions defined on $(0,+\infty)$ and $\mathcal{C}_b^1\big((0,+\infty)\big)$ be the set of continuously differentiable functions with bounded derivatives up to order 1. For simplicity, we denote by $\{P_\tau^{(\alpha,\lambda)},\ \tau \in (0,1)\}$ the Laguerre semigroup after the time change $-\log(\tau)/\lambda$ acting on functions in $\mathcal{B}\big((0,+\infty)\big)$. We recall the well-known transition kernel of this semigroup with respect to the Lebesgue measure:

$$p_\tau^{(\alpha,\lambda)}(x,u) = \frac{\lambda}{1-\tau}\left(\frac{u}{\tau x}\right)^{\frac{\alpha-1}{2}}\exp\left(-\frac{\lambda u}{1-\tau}\right)\exp\left(-\frac{\lambda\tau x}{1-\tau}\right)I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)$$

Moreover, it should be clear from the previous formula that the Laguerre semigroup is symmetric on $L^2\big(\mathbb{R}_+^*, \gamma_{\alpha,\lambda}(u)du\big)$.

Lemma 17. *Let $\alpha \geq \frac{1}{2}$. We have the following stochastic representations:*

$$\forall f \in \mathcal{B}\big((0,+\infty)\big),\ P_\tau^{(\alpha,\lambda)}(f)(x) = \mathbb{E}\left[f\left((1-\tau)\gamma(\alpha - \frac{1}{2},\lambda) + \left(\sqrt{\tau}\sqrt{x} + \sqrt{\frac{1-\tau}{2\lambda}}Z\right)^2\right)\right], \quad (8)$$

$$\forall f \in \mathcal{C}_b^1\big((0,+\infty)\big),\ \partial_x^\sigma\big(P_\tau^{(\alpha,\lambda)}(f)\big)(x) = \sqrt{\tau}\mathbb{E}\left[\frac{\big(\sqrt{\tau}\sqrt{x} + \sqrt{\frac{1-\tau}{2\lambda}}Z\big)}{(X_\tau^x)^{\frac{1}{2}}}\partial^\sigma(f)(X_\tau^x)\right], \quad (9)$$

with:

$$\partial_x^\sigma(\phi)(x) = \sqrt{x}\phi'(x),$$

$$X_\tau^x = (1-\tau)\gamma(\alpha - \frac{1}{2},\lambda) + \left(\sqrt{\tau}\sqrt{x} + \sqrt{\frac{1-\tau}{2\lambda}}Z\right)^2.$$

Proof. : Let f be in $\mathcal{B}\big((0,+\infty)\big)$. Let $x \in (0,+\infty)$. Let X be a random variable whose law is the Dirac measure at x. We denote by $g^{(\alpha,\lambda,x)}(\tau,u)$ the density of the gamma smart path built with such a X denoted by X_τ^x. By definition of the Laguerre semigroup and Definition 1, we readily have:

$$P_\tau^{(\alpha,\lambda)}(f)(x) = \int_0^{+\infty} f(u)g^{(\alpha,\lambda,x)}(\tau,u)du = \mathbb{E}[f(X_\tau^x)].$$

Formula (9) follows by standard computations. $\qquad\Box$

We note in particular that:

$$\Big| \frac{\big(\sqrt{\tau}\sqrt{x} + \sqrt{\frac{1-\tau}{2\lambda}}Z\big)}{(X_\tau^x)^{\frac{1}{2}}} \Big| \le 1 \tag{10}$$

Using (9), we want to derive a simple bound for the Fisher information structure along the evolute X_τ involving the Fisher information structure of X. From it, we obtain the logarithmic Sobolev inequality for the gamma law which is known to hold for $\alpha \ge 1/2$ ([3]). This approach extends a classical Cauchy-Schwarz argument from the Gaussian to the gamma case. Let f_X be the density of the random variable X.

Proposition 18. *Let $\alpha \ge \frac{1}{2}$. Assume that $J_{st,\gamma}(X) < +\infty$ and that f_X is smooth enough. Then, we have:*

$$D(X\|\gamma_{\alpha,\lambda}) \le \frac{1}{\lambda}J_{st,\gamma}(X).$$

Proof. The density of X_τ is given by:

$$g^{(\alpha,\lambda)}(\tau,u) = \int_0^{+\infty} p_\tau^{(\alpha,\lambda)}(x,u)f_X(x)dx.$$

By definition of the Fisher information structure, we have:

$$
\begin{aligned}
J_{st,\gamma}(X_\tau) &= \int_0^{+\infty} u g^{(\alpha,\lambda)}(\tau,u)\Big[\partial_u\big(\log(\frac{g^{(\alpha,\lambda)}(\tau,u)}{\gamma_{\alpha,\lambda}(u)}))\big)\Big]^2 du,\\
&= \int_0^{+\infty} u\frac{\big(\partial_u\big(f^{(\alpha,\lambda)}(\tau,u)\big)\big)^2}{f^{(\alpha,\lambda)}(\tau,u)}\gamma_{\alpha,\lambda}(u)du,
\end{aligned}
$$

with $f^{(\alpha,\lambda)}(\tau,u) = g^{(\alpha,\lambda)}(\tau,u)/\gamma_{\alpha,\lambda}(u)$. Using duality and symmetry of the transition kernel of the Laguerre semigroup with respect to the measure $\gamma_{\alpha,\lambda}(u)du$, we have the following representation:

$$J_{st,\gamma}(X_\tau) = \int_0^{+\infty} u\frac{\big(\partial_u\big(P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})\big)\big)^2}{P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})}\gamma_{\alpha,\lambda}(u)du.$$

At this point, we use the fact that f_X is smooth enough in order to ensure that (9) is true for the function $f_X/\gamma_{\alpha,\lambda}$. We obtain:

$$
\begin{aligned}
J_{st,\gamma}(X_\tau) &\le \tau \int_0^{+\infty} \frac{\mathbb{E}\big[\,|\,\partial^\sigma(f_X/\gamma_{\alpha,\lambda})(X_\tau^x)\,|\big]^2}{P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})}\gamma_{\alpha,\lambda}(u)du,\\
&\le \tau \int_0^{+\infty} \frac{P_\tau^{(\alpha,\lambda)}\big(\,|\,\partial^\sigma(f_X/\gamma_{\alpha,\lambda})\,|\big)^2}{P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})}\gamma_{\alpha,\lambda}(u)du,\\
&\le \tau \int_0^{+\infty} P_\tau^{(\alpha,\lambda)}\Big(\frac{|\,\partial^\sigma(f_X/\gamma_{\alpha,\lambda})\,|^2}{f_X/\gamma_{\alpha,\lambda}}\Big)\gamma_{\alpha,\lambda}(u)du,\\
&\le \tau \int_0^{+\infty} \frac{|\,\partial^\sigma(f_X/\gamma_{\alpha,\lambda})\,|^2}{f_X/\gamma_{\alpha,\lambda}}\gamma_{\alpha,\lambda}(u)du,\\
&\le \tau \int_0^{+\infty} u\frac{\big(\partial_u\big(f_X/\gamma_{\alpha,\lambda}\big)\big)^2}{f_X/\gamma_{\alpha,\lambda}}\gamma_{\alpha,\lambda}(u)du = \tau J_{st,\gamma}(X),
\end{aligned}
$$

where we have used successively (9), (10), Cauchy-Schwarz inequality and the invariance of the gamma measure along the Laguerre dynamic. Using De Bruijn identity (7) and integrating the previous

inequality, we obtain the following form of the logarithmic Sobolev inequality for the gamma measure $\gamma_{\alpha,\lambda}$ with $\alpha \geq 1/2$:

$$D(X\|\gamma_{\alpha,\lambda}) \leq \frac{1}{\lambda} J_{st,\gamma}(X).$$

$\square$

Remark 19. *Through the course of the previous proof, we have obtained the information theoretical inequality, $J_{st,\gamma}(X_\tau) \leq \tau J_{st,\gamma}(X)$, which should be compared to the following classical inequality obtained thanks to the Blachman-Stam inequality:*

$$\forall t \in (0,1), \ J_{st}(\sqrt{t}X + \sqrt{1-t}Z) \leq tJ_{st}(X),$$

where Z is a standard normal random variable independent of X and $J_{st}(.)$ denotes the classical standardized Fisher information structure associated with the Gaussian law.

6 A new HSI-type inequality

In this section, we develop the tools needed to obtain an HSI-type inequality linking relative Entropy, Stein discrepancy and standardized Fisher information in the spirit of the ones obtained in [25]. For this purpose, we define a new type of Stein kernel based on a rewriting of the Laguerre operator, $\mathcal{L}_{\alpha,\lambda}$. This rewriting is justified by the following lemma which provides Bismut-type representation for iterated actions of the operator ∂_x^σ previously introduced on the Laguerre semigroup.

Lemma 20. *Let $\alpha \geq 1/2$. Then, we have:*

$$\forall f \in \mathcal{B}\big((0,+\infty)\big), \ \partial_x^\sigma\big(P_\tau^{(\alpha,\lambda)}(f)\big)(x) = \sqrt{\frac{\lambda}{2}}\sqrt{\frac{\tau}{1-\tau}}\mathbb{E}\big[Zf(X_\tau^x)\big]. \tag{11}$$

Moreover, for any integer $k \geq 1$, we have:

$$\forall f \in \mathcal{B}\big((0,+\infty)\big), \ (\partial_x^\sigma)^k\big(P_\tau^{(\alpha,\lambda)}(f)\big)(x) = \left(\frac{\lambda}{2}\right)^{\frac{k}{2}}\left(\frac{\tau}{1-\tau}\right)^{\frac{k}{2}}\mathbb{E}\big[H_k(Z)f(X_\tau^x)\big],$$

where $H_k(.)$ denotes the k-th Hermite polynomial.

Proof. Let $f \in \mathcal{C}_b^1\big((0,+\infty)\big)$. By Lemma 17, we have:

$$\partial_x^\sigma\big(P_\tau^{(\alpha,\lambda)}(f)\big)(x) = \sqrt{\tau}\mathbb{E}\left[\frac{\big(\sqrt{\tau}\sqrt{x} + \sqrt{\frac{1-\tau}{2\lambda}}Z\big)}{(X_\tau^x)^{\frac{1}{2}}}\partial^\sigma(f)(X_\tau^x)\right],$$

$$= \sqrt{\tau}\mathbb{E}\left[\big(\sqrt{\tau}\sqrt{x} + \sqrt{\frac{1-\tau}{2\lambda}}Z\big)f'(X_\tau^x)\right],$$

$$= \sqrt{\frac{\lambda}{2}}\sqrt{\frac{\tau}{1-\tau}}\mathbb{E}\big[Zf(X_\tau^x)\big],$$

where we have performed a Gaussian integration by parts in order to get the last line. We proceed by density to extend this relation to functions in $\mathcal{B}\big((0,+\infty)\big)$. The general case is obtained by a recursive argument together with standard relations regarding Hermite polynomials. $\square$

Recall that the Laguerre operator is given by the following formula on sufficiently smooth functions:

$$\mathcal{L}_{\alpha,\lambda}(f)(u) = u\frac{d^2f}{du^2}(u) + (\alpha - \lambda u)\frac{df}{du}(u).$$

Using the operator ∂_x^σ this can be rewritten as:

$$\mathcal{L}_{\alpha,\lambda}(f)(u) = \big(\partial_u^\sigma\big)^2(f)(u) + (\alpha - \frac{1}{2} - \lambda u)\frac{df}{du}(u).$$

Thus, it is natural to introduce the following Stein kernel for probability measure on $(0,+\infty)$.

Definition 3. *Let X be a random variable with values in $(0, +\infty)$. Then, we define the Stein kernel of X, $\tau_X(.)$, for every smooth test function by:*

$$\mathbb{E}[(\lambda X - \alpha + \frac{1}{2})\phi(X)] = \mathbb{E}[\tau_X(X)\sqrt{X}(\partial_x^\sigma)^*(\phi)(X)],$$

where $(\partial_x^\sigma)^*(.) = \frac{d}{dx}(\sqrt{x}.)$. In particular, we have:

$$\mathbb{E}[(\lambda X - \alpha + \frac{1}{2})\phi'(X)] = \mathbb{E}[\tau_X(X)(\partial_x^\sigma)^2(\phi)(X)]. \tag{12}$$

Remark 21. *When $\mathbb{E}[X] = \alpha/\lambda$, we note that $\mathbb{E}[\tau_X(X)] = 1$. In particular, the Stein discrepancy is exactly the variance of $\tau_X(X)$.*

Before stating the main result of this section, we introduce a fundamental representation of the standardized Fisher information structure in terms of the previously defined Stein kernel. This representation should be compared to the one obtained in Proposition 2.4 (iii) of [25]. At the core of its proof stand the Bismut-type representation of $\partial_x^\sigma(P_\tau^{(\alpha,\lambda)}(f))$ as well as the intertwining relation (9). We assume that the random variable X has a density f_X such that $f = f_X/\gamma_{\alpha,\lambda}$ is smooth enough for the different analytical arguments to hold. Moreover, we assume that the Stein kernel of X exists.

Proposition 22. *Let $\alpha \geq 1/2$. We have:*

$$J_{st,\gamma}(X_\tau) = \sqrt{\frac{\lambda}{2}}\frac{\tau}{\sqrt{1-\tau}}\mathbb{E}[(\tau_X(X) - 1)Z\,\mathcal{V}(X_\tau)\partial^\sigma(v_\tau)(X_\tau)],$$

with

$$v_\tau = \log\left(P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})\right),$$

$$\mathcal{V}(X_\tau) = \frac{\left(\sqrt{\tau}\sqrt{X} + \sqrt{\frac{1-\tau}{2\lambda}}Z\right)}{\sqrt{(1-\tau)\gamma(\alpha - \frac{1}{2}, \lambda) + \left(\sqrt{\tau}\sqrt{X} + \sqrt{\frac{1-\tau}{2\lambda}}Z\right)^2}}.$$

Proof. Recall that we have the following representation for the Fisher information structure along the smart path:

$$J_{st,\gamma}(X_\tau) = \int_0^{+\infty} u\frac{\left(\partial_u\left(P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})\right)\right)^2}{P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})}\gamma_{\alpha,\lambda}(u)du. \tag{13}$$

Since the Laguerre generator is a diffusion, it satisfies the following integration by parts formula on smooth functions ϕ, ψ from $(0, +\infty)$ to $\mathbb{R}$:

$$\int_0^{+\infty} \phi(u)\mathcal{L}_{\alpha,\lambda}(\psi)(u)\gamma_{\alpha,\lambda}(u)du = -\int_0^{+\infty} \partial_u^\sigma(\phi)\partial_u^\sigma(\psi)(u)\gamma_{\alpha,\lambda}(u)du.$$

Thus, we have:

$$J_{st,\gamma}(X_\tau) = -\int_0^{+\infty} \mathcal{L}_{\alpha,\lambda}\left(P_\tau^{(\alpha,\lambda)}(v_\tau)\right)(x)f_X(x)dx,$$

$$= -\int_0^{+\infty} \left[(\partial_x^\sigma)^2(P_\tau^{(\alpha,\lambda)}(v_\tau))(x) + (\alpha - \frac{1}{2} - \lambda x)\partial_x(P_\tau^{(\alpha,\lambda)}(v_\tau))(x)\right]f_X(x)dx,$$

$$= -\int_0^{+\infty} \left[1 - \tau_X(x)\right](\partial_x^\sigma)^2(P_\tau^{(\alpha,\lambda)}(v_\tau))(x)f_X(x)dx,$$

where we have used the definition of the Stein kernel on the function $\partial_x(P_\tau^{(\alpha,\lambda)}(v_\tau))$. Now, thanks to Bismut formula (11), we have:

$$\partial_x^\sigma(P_\tau^{(\alpha,\lambda)}(v_\tau))(x) = \sqrt{\frac{\lambda}{2}}\sqrt{\frac{\tau}{1-\tau}}\mathbb{E}\left[Zv_\tau(X_\tau^x)\right].$$

Moreover, using the intertwining relation (9), we obtain:

$$(\partial_x^\sigma)^2(P_\tau^{(\alpha,\lambda)}(v_\tau))(x) = \sqrt{\frac{\lambda}{2}}\frac{\tau}{\sqrt{1-\tau}}\mathbb{E}\left[Z\frac{(\sqrt{\tau}\sqrt{x}+\sqrt{\frac{1-\tau}{2\lambda}}Z)}{(X_\tau^x)^{\frac{1}{2}}}\partial^\sigma(v_\tau)(X_\tau^x)\right].$$

The conclusion follows by integrating out with respect to the law of X. $\qquad\square$

We are now ready to state the main result of this section.

Theorem 23. *Let $\alpha \geq 1/2$. Let X be a strictly positive random variable with finite $\alpha + 4$ moments and such that $J_{st,\gamma}(X) < +\infty$, $\mathbb{E}[(\tau_X(X) - 1)^2] < +\infty$ and $\mathbb{E}[X] = \alpha/\lambda$. We have:*

$$D(X\|\gamma_{\alpha,\lambda}) \leq \frac{1}{2}\mathbb{E}[(\tau_X(X) - 1)^2]\log\left[1 + \frac{2}{\lambda}\frac{J_{st,\gamma}(X)}{\mathbb{E}[(\tau_X(X) - 1)^2]}\right].$$

Remark 24.   • *This inequality should be compared to the one of Proposition 4.3 in [25]. In particular, the Stein kernel used in the definition of the Stein discrepancy is different from the one we use here. In contrast to the proof of this result, we do not use the operators Γ_2 and Γ_3 from the Γ-calculus of [5]. This should emphasize the strength of stochastic representations such as the ones in Lemmata 17 and 20. The method of the proof is similar to the one of Theorem 2.2 of [25] in the Gaussian case.*

  • *Moreover, it is important to note that this functional inequality improves upon the classical logarithmic Sobolev inequality (Proposition 18) as in the Gaussian case.*

Proof. From De Bruijn identity of Theorem 15, we have:

$$\begin{aligned}D(X\|\gamma_{\alpha,\lambda}) &= \int_0^1 \frac{1}{\lambda\tau}J_{st,\gamma}(X_\tau)d\tau, \\ &= \int_0^u \frac{1}{\lambda\tau}J_{st,\gamma}(X_\tau)d\tau + \int_u^1 \frac{1}{\lambda\tau}J_{st,\gamma}(X_\tau)d\tau.\end{aligned}$$

For times closed to 1, we use the following bound (from the proof of Proposition 18):

$$\int_u^1 \frac{1}{\lambda\tau}J_{st,\gamma}(X_\tau)d\tau \leq \int_u^1 \frac{\tau}{\lambda\tau}J_{st,\gamma}(X)d\tau \leq \frac{1}{\lambda}J_{st,\gamma}(X)(1-u). \tag{14}$$

Moreover, thanks to Proposition 22, (10), and Cauchy-Schwarz inequality, we have:

$$J_{st,\gamma}(X_\tau) \leq \sqrt{\frac{\lambda}{2}}\frac{\tau}{\sqrt{1-\tau}}\mathbb{E}[(\tau_X(X)-1)^2]^{\frac{1}{2}}\mathbb{E}[|\partial^\sigma(v_\tau)(X_\tau)|^2]^{\frac{1}{2}}.$$

But,

$$\begin{aligned}\mathbb{E}[|\partial^\sigma(v_\tau)(X_\tau)|^2] &= \int_0^{+\infty} P_\tau^{(\alpha,\lambda)}(|\partial^\sigma(v_\tau)|^2)(x)f_X(x)dx, \\ &= \int_0^{+\infty} |\partial^\sigma(v_\tau)(u)|^2 P_\tau^{(\alpha,\lambda)}(f_X/\gamma_{\alpha,\lambda})(u)\gamma_{\alpha,\lambda}(u)du, \\ &= J_{st,\gamma}(X_\tau).\end{aligned}$$

21

Thus, we have the following bound:

$$J_{st,\gamma}(X_\tau) \leq \frac{\lambda}{2}\frac{\tau^2}{1-\tau}\mathbb{E}[(\tau_X(X)-1)^2],$$

which provides the following estimate for small times:

$$\int_0^u \frac{1}{\lambda\tau}J_{st,\gamma}(X_\tau)d\tau \leq \frac{1}{2}\mathbb{E}[(\tau_X(X)-1)^2]\int_0^u \frac{\tau}{1-\tau}d\tau,$$
$$\leq -\frac{1}{2}\mathbb{E}[(\tau_X(X)-1)^2](u+\log(1-u)).$$

The result follows with an optimisation in $u \in (0,1)$. $\qquad\square$

7 Appendix

Proposition 25. *Let $\alpha \geq 1/2$. We have for every $(\tau, u) \in (0,1) \times (0,+\infty)$:*

$$\partial_u\big(g^{(\alpha,\lambda)}\big)(\tau,u) = \frac{\alpha-1}{u}g^{(\alpha,\lambda)}(\tau,u) - \frac{\lambda}{1-\tau}g^{(\alpha,\lambda)}(\tau,u) + h^{(\alpha,\lambda)}(\tau,u), \qquad (15)$$

$$-\lambda\tau\frac{\partial g^{(\alpha,\lambda)}}{\partial\tau}(\tau,u) = -\frac{\partial}{\partial u}\left(g^{(\alpha,\lambda)}(\tau,u)u\frac{\partial}{\partial u}\log\left(\frac{g^{(\alpha,\lambda)}(\tau,u)}{\gamma_{\alpha,\lambda}(u)}\right)\right), \qquad (16)$$

$$= g^{(\alpha,\lambda)}(\tau,u)\left(u\frac{\lambda^2\tau}{(1-\tau)^2} - \frac{\lambda\alpha\tau}{1-\tau}\right) + h^{(\alpha,\lambda)}(\tau,u)\left(\alpha - u\lambda\frac{1+\tau}{1-\tau}\right)$$
$$+ uk^{(\alpha,\lambda)}(\tau,u), \qquad (17)$$

with,

$$h^{(\alpha,\lambda)}(\tau,u) = \frac{\lambda^2}{(1-\tau)^2}\big(\frac{1}{\tau}\big)^{\frac{\alpha-2}{2}}u^{\frac{\alpha-2}{2}}\exp\big(-\frac{\lambda}{1-\tau}u\big)$$
$$\times \int_0^{+\infty}\big(\frac{1}{x}\big)^{\frac{\alpha-2}{2}}\exp\big(-\frac{\lambda\tau}{1-\tau}x\big)I_\alpha\big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\big)\mathbb{P}_X(dx),$$

$$k^{(\alpha,\lambda)}(\tau,u) = \frac{\lambda^3}{(1-\tau)^3}\big(\frac{1}{\tau}\big)^{\frac{\alpha-3}{2}}u^{\frac{\alpha-3}{2}}\exp\big(-\frac{\lambda}{1-\tau}u\big)$$
$$\times \int_0^{+\infty}\big(\frac{1}{x}\big)^{\frac{\alpha-3}{2}}\exp\big(-\frac{\lambda\tau}{1-\tau}x\big)I_{\alpha+1}\big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\big)\mathbb{P}_X(dx).$$

Proof. We begin by computing the first partial derivative of $g^{(\alpha,\lambda)}(.,.)$ with respect to u. In order to do so, we need to justify properly the interchange of derivative and integral since:

$$g^{(\alpha,\lambda)}(\tau,u) = \frac{\lambda}{1-\tau}\left(\frac{u}{\tau}\right)^{\frac{\alpha-1}{2}}\exp\left(-\frac{\lambda u}{1-\tau}\right)\int_0^{+\infty}\big(\frac{1}{x}\big)^{\frac{\alpha-1}{2}}\exp\left(-\frac{\lambda\tau x}{1-\tau}\right)I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx).$$

Let $K = [a,b]$ be a compact set strictly contained in $(0,+\infty)$. We need to control uniformly the following quantity for $u \in K$:

$$(I) = |\big(\frac{1}{x}\big)^{\frac{\alpha-1}{2}}\exp\left(-\frac{\lambda\tau x}{1-\tau}\right)\partial_u\big(I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\big)|$$

Using the fact that $I_{\alpha-1}^{(1)}(z) = I_\alpha(z) + I_{\alpha-1}(z)(\alpha-1)/z$, we obtain the following straightforward bound:

$$(I) \leq (\frac{1}{x})^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) \frac{\lambda\sqrt{x\tau}}{(1-\tau)\sqrt{u}} \{I_\alpha\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)$$
$$+ \frac{|\alpha-1|(1-\tau)}{2\lambda\sqrt{ux\tau}} I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\},$$
$$\leq C_{K,\tau,\alpha,\lambda}\{(\frac{1}{x})^{\frac{\alpha-2}{2}} \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) I_\alpha\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)$$
$$+ (\frac{1}{x})^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\}.$$

In order to deal with the terms involving the modified Bessel functions of the first kind, we use the right-hand side of inequality (6.25) from [27] which holds for $\alpha > 1/2$. We obtain:

$$(I) \leq C'_{K,\tau,\alpha,\lambda}\{x \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) \cosh\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right) + \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) \cosh\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\},$$
$$\leq C'_{K,\tau,\alpha,\lambda}\{x \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) \cosh\left(\frac{2\lambda\sqrt{bx\tau}}{1-\tau}\right) + \exp\left(-\frac{\lambda\tau x}{1-\tau}\right) \cosh\left(\frac{2\lambda\sqrt{bx\tau}}{1-\tau}\right)\}.$$

The previous bound is clearly integrable with respect to the probability measure $\mathbb{P}_X$ on $(0, +\infty)$. We deal with the case $\alpha = 1/2$ using the explicit expression for the modified Bessel functions of the first kind of order $-1/2$. We can thus perform the computations for the first partial derivative of $g^{(\alpha,\lambda)}(.,.)$ with respect to u. We have:

$$\partial_u(g^{(\alpha,\lambda)})(\tau, u) = \frac{\lambda}{1-\tau}(\frac{1}{\tau})^{\frac{\alpha-1}{2}}(\frac{\alpha-1}{2})u^{\frac{\alpha-1}{2}-1} \exp\left(-\frac{\lambda}{1-\tau}u\right)$$
$$\times \int_0^{+\infty} (\frac{1}{x})^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau}{1-\tau}x\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx)$$
$$- \frac{\lambda}{1-\tau}\frac{\lambda}{1-\tau}(\frac{1}{\tau})^{\frac{\alpha-1}{2}}u^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda}{1-\tau}u\right)$$
$$\times \int_0^{+\infty} (\frac{1}{x})^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau}{1-\tau}x\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx)$$
$$+ \frac{\lambda}{1-\tau}(\frac{1}{\tau})^{\frac{\alpha-1}{2}}u^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda}{1-\tau}u\right)$$
$$\times \int_0^{+\infty} (\frac{1}{x})^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau}{1-\tau}x\right) \frac{2\lambda\sqrt{x\tau}}{1-\tau}\frac{1}{2\sqrt{u}} I_{\alpha-1}^{(1)}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx),$$
$$= \frac{\alpha-1}{2u}g^{(\alpha,\lambda)}(\tau, u) - \frac{\lambda}{1-\tau}g^{(\alpha,\lambda)}(\tau, u)$$
$$+ \frac{\alpha-1}{2u}\frac{\lambda}{1-\tau}(\frac{1}{\tau})^{\frac{\alpha-1}{2}}u^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda}{1-\tau}u\right)$$
$$\times \int_0^{+\infty} (\frac{1}{x})^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda\tau}{1-\tau}x\right) I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx)$$
$$+ \frac{\lambda^2}{(1-\tau)^2}(\frac{1}{\tau})^{\frac{\alpha-2}{2}}u^{\frac{\alpha-2}{2}} \exp\left(-\frac{\lambda}{1-\tau}u\right)$$
$$\times \int_0^{+\infty} (\frac{1}{x})^{\frac{\alpha-2}{2}} \exp\left(-\frac{\lambda\tau}{1-\tau}x\right) I_\alpha\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx).$$

where we have used the fact that $I_{\alpha-1}^{(1)}(z) = I_\alpha(z) + I_{\alpha-1}(z)(\alpha-1)/z$. Therefore, we have:

$$\partial_u\left(g^{(\alpha,\lambda)}\right)(\tau, u) = \frac{\alpha-1}{u}g^{(\alpha,\lambda)}(\tau, u) - \frac{\lambda}{1-\tau}g^{(\alpha,\lambda)}(\tau, u) + h^{(\alpha,\lambda)}(\tau, u).$$

By a similar reasoning, we obtain relations (16) and (17). $\square$

Lemma 26. *Let $\alpha > 0$. For $\tau \in (0,1)$, we have:*

$$g^{(\alpha,\lambda)}(\tau, u) \underset{u \to 0^+}{\sim} C_{\tau,\alpha,\lambda} u^{\alpha-1},$$

where $C_{\tau,\alpha,\lambda}$ is some strictly positive constant.

Proof. Let $\mu > 0$. The Laplace transform of $g^{(\alpha,\lambda)}(\tau,.)$ is given by:

$$L(g^{(\alpha,\lambda)}(\tau,.))(\mu) = \frac{1}{\left(1 + \frac{\mu}{\lambda}(1-\tau)\right)^\alpha} L(\mathbb{P}_X)\left(\frac{\mu\tau}{1 + \frac{\mu}{\lambda}(1-\tau)}\right).$$

Since $L(\mathbb{P}_X)$ is in $C^0(]0, +\infty[)$, we have:

$$L(g^{(\alpha,\lambda)}(\tau,.))(\mu) \underset{\mu \to +\infty}{\sim} \frac{\lambda^\alpha}{(1-\tau)^\alpha} L(\mathbb{P}_X)\left(\frac{\lambda\tau}{1-\tau}\right)\frac{1}{\mu^\alpha}.$$

Thus, by a classical Tauberian Theorem (see Chapter XIII.5, Theorem 3 of [20]), we have:

$$g^{(\alpha,\lambda)}(\tau, u) \underset{u \to 0^+}{\sim} \frac{\lambda^\alpha}{(1-\tau)^\alpha} L(\mathbb{P}_X)\left(\frac{\lambda\tau}{1-\tau}\right)\frac{u^\alpha}{\Gamma(\alpha+1)},$$

where $g^{(\alpha,\lambda)}(\tau,.)$ is the cumulative distribution function of X_τ. In order to conclude, we need to know if $g^{\alpha,\lambda}(\tau,.)$ is monotone in a right neighborhood of 0. Note that monotony properties of $g^{\alpha,\lambda}(\tau,.)$ can be deduced from those of $g^{(\alpha,\lambda)}(\tau,x,.)$. Moreover, since $I_\nu(z) \underset{0^+}{\sim} (z/2)^\nu 1/\Gamma(\nu+1)$, we have:

$$g^{(\alpha,\lambda)}(\tau, x, u) \underset{u \to 0^+}{\sim} C_{\tau,\alpha,\lambda,x} u^{\alpha-1} \exp\left(\frac{-\lambda u}{1-\tau}\right),$$

for some constant $C_{\tau,\alpha,\lambda,x} > 0$. This implies that $g^{\alpha,\lambda}(\tau,.)$ is monotone in a right neighborhood of 0. Thus,

$$g^{\alpha,\lambda}(\tau, u) \underset{u \to 0^+}{\sim} \frac{\alpha\lambda^\alpha}{(1-\tau)^\alpha} L(\mathbb{P}_X)\left(\frac{\lambda\tau}{1-\tau}\right)\frac{u^{\alpha-1}}{\Gamma(\alpha+1)},$$

which concludes the proof. $\qquad\square$

Lemma 27. *Let $\alpha \geq 1/2$. Let X be an almost surely positive random variable with first and second moments finite. Then, we have, for every $(\tau, u) \in (a,b) \times (0, +\infty)$:*

$$g^{(\alpha,\lambda)}(\tau, u) \leq C^1_{a,b,\lambda,\alpha}\left(u^{\alpha-1}\exp\left(-\lambda\frac{u}{4(1-a)}\right) + u^{\alpha-1}\int_{\frac{u}{4}}^{+\infty} \mathbb{P}_X(dx)\right),$$

$$h^{(\alpha,\lambda)}(\tau, u) \leq C^2_{a,b,\lambda,\alpha}\left(u^{\alpha}\exp\left(-\lambda\frac{u}{4(1-a)}\right) + u^{\alpha-1}\mathbb{E}[X\mathbb{I}_{\{X > \frac{u}{4}\}}]\right),$$

$$k^{(\alpha,\lambda)}(\tau, u) \leq C^3_{a,b,\lambda,\alpha}\left(u^{\alpha+1}\exp\left(-\lambda\frac{u}{4(1-a)}\right) + u^{\alpha-1}\mathbb{E}[X^2\mathbb{I}_{\{X > \frac{u}{4}\}}]\right),$$

with $C^i_{a,b,\lambda,\alpha}$, $i \in \{1,2,3\}$, some strictly positive constants.

Proof. By definition, for every $(\tau, u) \in (a,b) \times (0, +\infty)$, we have:

$$g^{(\alpha,\lambda)}(\tau, u) = \frac{\lambda}{1-\tau}\left(\frac{1}{\tau}\right)^{\frac{\alpha-1}{2}} u^{\frac{\alpha-1}{2}}\exp\left(-\frac{\lambda}{1-\tau}u\right)$$
$$\times \int_0^{+\infty}\left(\frac{1}{x}\right)^{\frac{\alpha-1}{2}}\exp\left(-\frac{\lambda\tau}{1-\tau}x\right)I_{\alpha-1}\left(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\right)\mathbb{P}_X(dx)$$

By inequality (6.25) in [27] which holds for $\alpha > 1/2$ (for $\alpha = 1/2$, we use the explicit expression of $I_{-1/2}(.)$), we have the following estimate:

$$I_{\alpha-1}\Big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\Big) \leq \frac{1}{2^{\alpha-1}\Gamma(\alpha)}\Big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\Big)^{\alpha-1} \cosh\Big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\Big),$$

$$\leq C_{a,b,\lambda,\alpha}(ux)^{\frac{\alpha-1}{2}} \exp\Big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\Big).$$

Thus, we have:

$$g^{(\alpha,\lambda)}(\tau,u) \leq C^1_{a,b,\lambda,\alpha} u^{\alpha-1} \int_0^{+\infty} \exp\Big(-\frac{\lambda}{1-\tau}(\sqrt{u}-\sqrt{x\tau})^2\Big)\mathbb{P}_X(dx),$$

$$\leq C^1_{a,b,\lambda,\alpha} u^{\alpha-1}\Big(\int_0^{\frac{u}{4\tau}} \exp\Big(-\frac{\lambda}{1-\tau}(\sqrt{u}-\sqrt{x\tau})^2\Big)\mathbb{P}_X(dx)$$

$$+ \int_{\frac{u}{4\tau}}^{+\infty} \exp\Big(-\frac{\lambda}{1-\tau}(\sqrt{u}-\sqrt{x\tau})^2\Big)\mathbb{P}_X(dx)\Big),$$

$$\leq C^1_{a,b,\lambda,\alpha} u^{\alpha-1}\Big(\exp\Big(-\frac{\lambda}{1-\tau}\frac{u}{4}\Big) + \int_{\frac{u}{4}}^{+\infty} \mathbb{P}_X(dx)\Big).$$

We proceed similarly for the functions $h^{(\alpha,\lambda)}(\tau,u)$ and $k^{(\alpha,\lambda)}(\tau,u)$. $\qquad\square$

Lemma 28. *Let $\alpha \geq 1/2$. There exists a strictly positive constant $A_{X,\alpha,\lambda,a,b}$ such that:*

$$\big|\log\big(g^{(\alpha,\lambda)}(\tau,u)\big)\big| \leq A_{X,\alpha,\lambda,a,b} + 2\frac{\lambda u}{1-b} + |\alpha-1|\big|\log(u)\big|.$$

Proof. Let $(a,b) \subsetneq (0,1)$. For every $(\tau,u) \in (a,b) \times (0,+\infty)$, we have:

$$g^{(\alpha,\lambda)}(\tau,u) = \frac{\lambda}{1-\tau}\Big(\frac{1}{\tau}\Big)^{\frac{\alpha-1}{2}} u^{\frac{\alpha-1}{2}} \exp\Big(-\frac{\lambda}{1-\tau}u\Big)$$

$$\times \int_0^{+\infty} \Big(\frac{1}{x}\Big)^{\frac{\alpha-1}{2}} \exp\Big(-\frac{\lambda\tau}{1-\tau}x\Big) I_{\alpha-1}\Big(\frac{2\lambda\sqrt{ux\tau}}{1-\tau}\Big)\mathbb{P}_X(dx).$$

From the previous lemma, we clearly have:

$$g^{(\alpha,\lambda)}(\tau,u) \leq C^1_{a,b,\lambda,\alpha} u^{\alpha-1}.$$

Moreover, using the following inequality (see the left-hand side of inequality (6.25) in [27]):

$$\forall \nu > -\frac{1}{2}, \ \forall z > 0, \ I_\nu(z) > \frac{z^\nu}{2^\nu}\frac{1}{\Gamma(\nu)},$$

we obtain:

$$g^{(\alpha,\lambda)}(\tau,u) \geq C_{X,a,b,\lambda,\alpha} u^{\alpha-1} \exp\Big(-\frac{\lambda u}{1-b}\Big).$$

Thus,

$$C_{X,a,b,\lambda,\alpha} u^{\alpha-1} \exp\Big(-\frac{\lambda u}{1-b}\Big) \leq g^{(\alpha,\lambda)}(\tau,u) \leq C^1_{a,b,\lambda,\alpha} u^{\alpha-1}.$$

Consequently, we obtain:

$$\big|\log\big(g^{(\alpha,\lambda)}(\tau,u)\big)\big|$$

$$\leq \big|\log\big(\frac{g^{(\alpha,\lambda)}(\tau,u)}{C_{X,a,b,\lambda,\alpha}u^{\alpha-1}\exp\big(-\frac{\lambda u}{1-b}\big)}\big)\big| + \big|\log\big(C_{X,a,b,\lambda,\alpha}u^{\alpha-1}\exp\big(-\frac{\lambda u}{1-b}\big)\big)\big|,$$

$$\leq \big|\log\big(\frac{C^1_{a,b,\lambda,\alpha}u^{\alpha-1}}{C_{X,a,b,\lambda,\alpha}u^{\alpha-1}\exp\big(-\frac{\lambda u}{1-b}\big)}\big)\big| + |\log(C_{X,a,b,\lambda,\alpha})| + |\alpha-1||\log(u)| + \frac{\lambda u}{1-b},$$

$$\leq |\log(C^1_{a,b,\lambda,\alpha})| + 2|\log(C_{X,a,b,\lambda,\alpha})| + |\alpha-1||\log(u)| + 2\frac{\lambda u}{1-b}.$$

The result then follows. $\qquad\square$

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