

GENERALIZED PASCAL TRIANGLE FOR BINOMIAL COEFFICIENTS OF WORDS

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ABSTRACT. We introduce a generalization of Pascal triangle based on binomial coefficients of finite words. These coefficients count the number of times a word appears as a subsequence of another finite word. Similarly to the Sierpiński gasket that can be built as the limit set, for the Hausdorff distance, of a convergent sequence of normalized compact blocks extracted from Pascal triangle modulo 2, we describe and study the first properties of the subset of $[0, 1] \times [0, 1]$ associated with this extended Pascal triangle modulo a prime p .

1. INTRODUCTION

Pascal triangle and the corresponding Sierpiński gasket are well studied objects (see, for instance, [15] for a survey). They exhibit self-similarity features and have connections with dynamical systems, cellular automata, number theoretic questions and automatic sequences [1, 2, 5, 10, 13]. In this paper, we will consider a variation of these two objects by extending binomial coefficients to the free monoid A^* where A is a finite alphabet.

Let us start with basic combinatorial definitions. A *finite word* is simply a finite sequence of letters belonging to a finite set called *alphabet*. In combinatorics on words, one can introduce the binomial coefficient $\binom{u}{v}$ of two finite words u and v which is the number of times v occurs as a subsequence of u (meaning as a “scattered” subword). As an example, we consider two particular words over $\{0, 1\}$ and

$$\binom{101001}{101} = 6.$$

Indeed, if we index the letters of the first word $u_1u_2 \cdots u_6 = 101001$, we have

$$u_1u_2u_3 = u_1u_2u_6 = u_1u_4u_6 = u_1u_5u_6 = u_3u_4u_6 = u_3u_5u_6 = 101.$$

Observe that this concept is a natural generalization of the binomial coefficients of integers. For a single letter alphabet $\{a\}$, we have

$$(1) \quad \binom{a^m}{a^n} = \binom{m}{n}, \quad \forall m, n \in \mathbb{N}$$

where a^m denotes the concatenation of m letters a . For more on these binomial coefficients, see for instance [11, Chap. 6].

In this paper, we are interested in Pascal triangle obtained when considering binomial coefficients of words. To define such a triangular array, we will consider all the words over a finite alphabet and we order them by genealogical ordering

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(i.e., first by length, then by the classical lexicographic ordering for words of the same length assuming $0 < 1$). For the sake of simplicity, we will mostly discuss the case of a 2-letter alphabet $\{0, 1\}$ and to relate these words to base-2 expansions of integers, we will assume without loss of generality that the non-empty words start with 1. (If leading zeroes were allowed, then different words could represent the same integer.)

Definition 1. We let $\text{rep}_2(n)$ denote the greedy base-2 expansion of $n \in \mathbb{N}_{>0}$ starting with 1 where the notation $\mathbb{N}_{>0}$ stands for the set of all positive integers. We set $\text{rep}_2(0)$ to be the empty word denoted by ε . Let $L = \{\varepsilon\} \cup 1\{0, 1\}^*$ be the set of base-2 expansions of the integers. We let L_n denote the set of words of length at most n belonging to L , i.e.,

$$L_n = (\{\varepsilon\} \cup 1\{0, 1\}^*) \cap \{0, 1\}^{\leq n}.$$

Note that $\#L_n = 2^n$ for all $n \geq 0$.

The first few values in the generalized Pascal triangle T that we will deal with are given in Table 1. These values correspond to the words $\varepsilon, 1, 10, 11, 100, 101, 110, 111$.

	ε	1	10	11	100	101	110	111
ε	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
10	1	1	1	0	0	0	0	0
11	1	2	0	1	0	0	0	0
100	1	1	2	0	1	0	0	0
101	1	2	1	1	0	1	0	0
110	1	2	2	1	0	0	1	0
111	1	3	0	3	0	0	0	1

TABLE 1. The first few values in the generalized Pascal triangle.

Considering the intersection of the lattice \mathbb{N}^2 with the region $[0, 2^j] \times [0, 2^j]$, the first 2^j rows and columns of the usual Pascal triangle $(\binom{m}{n} \bmod 2)_{m, n < 2^j}$ provide a coloring of this lattice. If we normalize this region by a homothety of ratio $1/2^j$, we get a sequence in $[0, 1] \times [0, 1]$ converging, for the Hausdorff distance, to the Sierpiński gasket when j tends to infinity [10].

It is therefore natural to ask whether a similar phenomenon occurs in our extended framework. The generalized Pascal triangle limited to words in L_n has 2^n rows and columns. After coloring and normalization (see Figure 1 for a picture of the cases when $n = 3, 4$ and Figure 10 in the appendix for $n = 9$), can we expect the convergence to an analogue of the Sierpiński gasket and could we describe the possible limit object? We answer positively and the limit object is described as the topological closure of a union of segments that are described through a simple combinatorial property (see Figure 9 in the appendix). For the sake of simplicity, we will mostly describe the coloring modulo 2. At the end of the paper, we shortly discuss colorings modulo a prime number p (see Figure 11 for coefficients congruent to 2 modulo 3).

In our construction, we take at each step exactly 2^n words and a scaling (or normalization) factor of $1/2^n$. For instance, the authors in [3] discussed which

sequences can be used as scaling factors for objects related to automatic sequences. In particular, the classical Pascal triangle modulo p^d is shown to be p -automatic in [1], where p is prime and the scaling sequence has to be of the form $(p^{kn+j})_{n \geq 0}$ with $j = 0, \dots, p - 1$.

Remark 2. We can make some extra comments about our study. Note that, from (1) the usual Pascal triangle is a “sub-array” of our extended triangle T by considering only words of the form 1^m and 1^n . In Table 1, the elements of the usual triangle are written in bold. We can also observe that the second column of this extended triangle T is exactly $(s_2(n))_{n \geq 0}$ where s_2 denotes the sum-of-digits function for base-2 expansions of integers [6]. Thus, considering these values modulo 2, the second column is exactly the well-known Thue–Morse word, see [4] for connections between binomial coefficients of words and p -adic topology. Moreover, each column of T is related to the language (i.e., the set of finite words)

$$L(v, r, p) := \left\{ u \in A^* \mid \binom{u}{v} \equiv r \pmod{p} \right\}, \quad v \in A^*, r \in \{0, \dots, p - 1\}.$$

In formal language theory [7], a language is a p -group language if and only if it is a Boolean combination of languages $L(v_i, r_j, p)$. For an extension of a theorem of Mahler in p -adic analysis to functions defined on the free monoid A^* , see [14].

This paper is organized as follows. In Section 2, we collect some results on binomial coefficients. In Section 3, we define a subset T_n of $[0, 2^n] \times [0, 2^n]$ associated with the parity of the first 2^n binomial coefficients. We prove that the subset of non-zero binomial coefficients for words in L_n contains exactly 3^n elements. The set T_n is then normalized by a factor $1/2^n$ to give the sequence $(U_n)_{n \geq 0}$ of subsets in $[0, 1] \times [0, 1]$. In the second part of Section 3, we independently build a compact set \mathcal{A}_0 from a countable union of segments and an important combinatorial condition (\star) . This condition is at the heart of our discussions (and will be generalized in Section 5 to take into account the situation modulo p). From \mathcal{A}_0 and using two maps h and c , the first one being a homothety, we define a sequence of compact sets naturally converging to a limit compact set \mathcal{L} . In Section 4, similarly to the construction of the Sierpiński gasket, we prove that $(U_n)_{n \geq 0}$ tends to \mathcal{L} . For the sake of simplicity, we have limited our presentation to the case of odd binomial coefficients. In Section 5, we briefly sketch the main changes when considering congruences modulo a prime p . In the last section, we present some pictures of the studied sets. The limit set \mathcal{L} convinces us that these extended Pascal triangles contain many interesting combinatorial and dynamical questions to consider.

2. BASIC RESULTS ON BINOMIAL COEFFICIENTS

In the first part of this section, we collect well-known facts on binomial coefficients of words or integers. For a proof of the first two lemmas, we refer the reader to [11, Chap. 6].

Lemma 3. *Let u, v be two words of L and let a, b be two letters in $\{0, 1\}$. Then we have*

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b} \binom{u}{v}$$

where $\delta_{a,b}$ is equal to 1 if $a = b$, 0 otherwise.

Lemma 4. *Let s, t, w be three words of L . Then we have*

$$\binom{sw}{t} = \sum_{\substack{u, v \in L \\ uv=t}} \binom{s}{u} \binom{w}{v}.$$

The main observation of the next lemma is that if $\binom{u}{vb} \equiv 1 \pmod{2}$ for some $b \in \{0, 1\}$, then there exists $a \in \{0, 1\}$ such that $\binom{ua}{vb} \equiv 1 \pmod{2}$.

Lemma 5. *Let u, v be words of L . The following tables display the values modulo 2 of the binomial coefficient $\binom{ua}{vb}$ for all $a, b \in \{0, 1\}$ when the values modulo 2 of the binomial coefficients $\binom{u}{v}$ and $\binom{u}{vb}$ with $b \in \{0, 1\}$ are known.*

	v	$v0$	$v1$		v	$v0$	$v1$		v	$v0$	$v1$		v	$v0$	$v1$
u	0	0	0	u	0	0	1	u	0	1	0	u	0	1	1
$u0$		0	0	$u0$		0	1	$u0$		1	0	$u0$		1	1
$u1$		0	0	$u1$		0	1	$u1$		1	0	$u1$		1	1
	v	$v0$	$v1$		v	$v0$	$v1$		v	$v0$	$v1$		v	$v0$	$v1$
u	1	0	0	u	1	0	1	u	1	1	0	u	1	1	1
$u0$		1	0	$u0$		1	1	$u0$		0	0	$u0$		0	1
$u1$		0	1	$u1$		0	0	$u1$		1	1	$u1$		1	0

Proof. It directly follows from Lemma 3. □

Let us also recall Lucas' theorem relating classical binomial coefficients modulo a prime p with base- p expansions. See [12, p. 230] or [9]. Note that in the following statement, if the base- p expansions of m and n are not of the same length, then we pad the shorter with leading zeroes.

Theorem 6. *Let m and n be two non-negative integers and let p be a prime. If*

$$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0$$

and

$$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$$

with $m_i, n_i \in \{0, \dots, p-1\}$ for all i , then the following congruence relation holds

$$\binom{m}{n} \equiv \prod_{i=0}^k \binom{m_i}{n_i} \pmod{p},$$

using the following convention: $\binom{m}{n} = 0$ if $m < n$.

3. GRAPHICAL REPRESENTATION

We let w_i denote the i th word of the language L in the genealogical order. Note that $w_i = \text{rep}_2(i)$ for $i \geq 0$. Considering the intersection of the lattice \mathbb{N}^2 with the region $[0, 2^n] \times [0, 2^n]$, the first 2^n rows and columns of the generalized Pascal triangle

$$\left(\binom{w_i}{w_j} \pmod{2} \right)_{0 \leq i, j < 2^n}$$

provide a coloring of this lattice leading to the definition of a sequence $(T_n)_{n \geq 0}$ of subsets of \mathbb{R}^2 . If we normalize these T_n by a homothety of ratio $1/2^n$, we will define a sequence $(U_n)_{n \geq 0}$ in $[0, 1] \times [0, 1]$.

3.1. **Definition of the sequences $(T_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$.**

Definition 7. Let $Q := [0, 1] \times [0, 1]$. Consider the sequence $(T_n)_{n \geq 0}$ of sets in \mathbb{R}^2 defined for all $n \geq 0$ by

$$T_n := \bigcup \left\{ (\text{val}_2(v), \text{val}_2(u)) + Q \mid u, v \in L_n, \binom{u}{v} \equiv 1 \pmod{2} \right\} \subset [0, 2^n] \times [0, 2^n].$$

Each T_n is a finite union of unit squares and is thus compact.

Remark 8. In Table 2, we count the number of unit squares in T_n for the first few values of n and we compare this quantity to the number of positive binomial coefficients of pairs of words in L_n . Observe that, as shown in the next lemma, the

	1	2	3	4	5	6	7	8	9	10
# unit squares	3	8	22	62	166	458	1258	3510	9838	27598
# positive coefficients	3	9	27	81	243	729	2187	6561	19683	59049

TABLE 2. Number of unit squares in T_n , $n = 1, \dots, 10$.

number of positive binomial coefficients of pairs of words in L_n is equal to 3^n .

Lemma 9. *For all $n \in \mathbb{N}$, the number of pairs of words in L_n having a positive binomial coefficient is equal to 3^n .*

Proof. For any positive integer n , let V_n denote the set of pairs of integers (x, y) such that $2^{n-1} \leq y < 2^n$, $0 \leq x \leq y$ and $\binom{\text{rep}_2(y)}{\text{rep}_2(x)} > 0$. Thus, $\#V_n$ corresponds to the number of pairs of words in $(L_n \setminus L_{n-1}) \times L_n$ having a positive binomial coefficient. We prove that $\#V_n = 2 \cdot 3^{n-1}$ by induction on $n \geq 1$. This proves the result since the number of positive binomial coefficients of pairs of words in L_0 is $1 = 3^0$.

The result is clear for $n = 1$. Let us suppose it is true up to n and let us prove it for $n + 1$. For all integers $n \geq 1$ and $m \geq 0$, consider the set

$$X_{m,n} := \begin{cases} \emptyset, & \text{if } m > n; \\ V_n \cap (\{0\} \times \mathbb{N}), & \text{if } m = 0; \\ V_n \cap ([2^{m-1}, 2^m] \times \mathbb{N}), & \text{otherwise.} \end{cases}$$

We thus have the following partition

$$V_n = \bigcup_{m=0}^n X_{m,n},$$

with, for $m \geq 1$,

$$X_{m,n} = \left\{ (\text{val}_2(v), \text{val}_2(u)) \mid u \in L_n \setminus L_{n-1}, v \in L_m \setminus L_{m-1}, \binom{u}{v} > 0 \right\}.$$

Let us define the functions f_1, f_2, f_3 and f_4 by

$$\begin{aligned} f_1(x, y) &= (2x, 2y), \\ f_2(x, y) &= (2x + 1, 2y + 1), \\ f_3(x, y) &= (x, 2y), \\ f_4(x, y) &= (x, 2y + 1). \end{aligned}$$

It follows from Lemma 3 that for all $n \geq 1$ and $m \geq 0$,

$$X_{m+1,n+1} = f_1(X_{m,n}) \cup f_2(X_{m,n}) \cup f_3(X_{m+1,n}) \cup f_4(X_{m+1,n}).$$

We thus get

$$\begin{aligned} V_{n+1} &= \bigcup_{m=0}^{n+1} X_{m,n+1} \\ &= (\{0\} \times ([2^n, 2^{n+1}) \cap \mathbb{N})) \\ &\quad \cup \left(\bigcup_{m=0}^n f_1(X_{m,n}) \cup f_2(X_{m,n}) \cup f_3(X_{m+1,n}) \cup f_4(X_{m+1,n}) \right) \\ &= f_1(V_n) \cup f_2(V_n) \cup f_3(V_n) \cup f_4(V_n). \end{aligned}$$

Observe that $f_1(V_n) \cap f_2(V_n) = \emptyset$ and $f_3(V_n) \cap f_4(V_n) = \emptyset$. Furthermore, if $(x, y) \in V_n$, then exactly one of the two elements $f_3(x, y)$ and $f_4(x, y)$ belongs to $f_1(V_n) \cup f_2(V_n)$. Indeed, if (x, y) belongs to V_n , then so does $(\lfloor x/2 \rfloor, y)$ and exactly one of the following two equalities is satisfied (depending on the parity of x):

$$f_3(x, y) = f_1(\lfloor x/2 \rfloor, y) \quad \text{or} \quad f_4(x, y) = f_2(\lfloor x/2 \rfloor, y).$$

We thus get $\#V_{n+1} = 3\#V_n = 2 \cdot 3^n$, which concludes the proof. \square

Definition 10. We will be interested in the sequence $(U_n)_{n \geq 0}$ of compact sets defined for all $n \geq 0$ by

$$U_n := \frac{T_n}{2^n} \subset [0, 1] \times [0, 1].$$

In Figure 1, we have depicted the sets U_3 and U_4 . The set U_9 is depicted in Figure 10 given in the appendix.

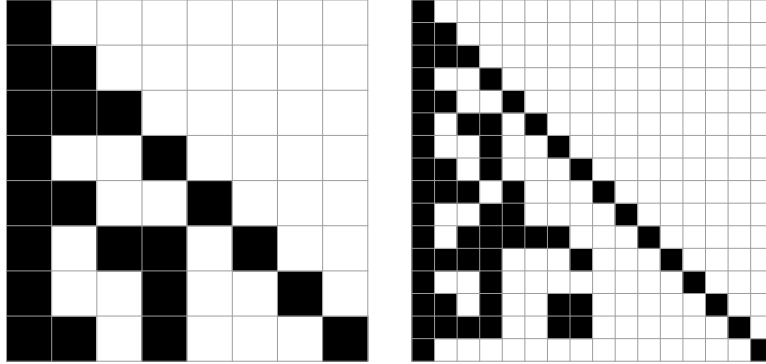


FIGURE 1. The sets U_3 and U_4 .

If $u = u_1 \cdots u_n$ is a finite word over $\{0, 1\}$, we make use of the following convention: $0.u$ has to be understood as the rational number $\sum_{i=1}^n u_i/2^i$.

Remark 11. Each pair (u, v) of words of length at most n with an odd binomial coefficient gives rise to a square region in U_n . More precisely, we have the following situation. Let $n \geq 0$ and $u, v \in L_n$ such that $\binom{u}{v} \equiv 1 \pmod{2}$. We have

$$(0.0^{n-|v|}v, 0.0^{n-|u|}u) + Q/2^n \subset U_n$$

as depicted in Figure 2.

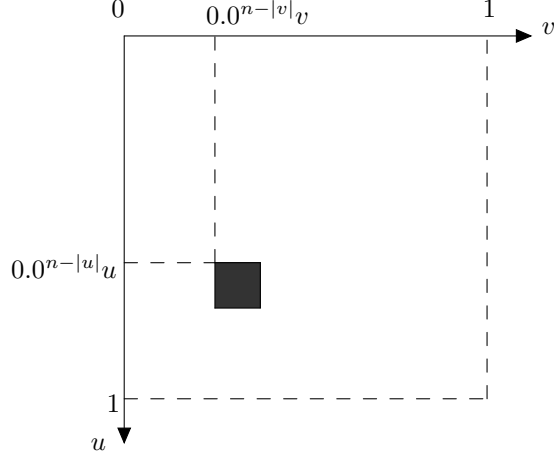


FIGURE 2. Visualization of a square region in U_n .

We will consider the space $(\mathcal{H}(\mathbb{R}^2), d_h)$ of the non-empty compact subsets of \mathbb{R}^2 equipped with the Hausdorff metric d_h induced by the Euclidean distance d on \mathbb{R}^2 . It is well known that $(\mathcal{H}(\mathbb{R}^2), d_h)$ is complete [8]. We let $B(x, \epsilon)$ denote the open ball of radius ϵ centered at $x \in \mathbb{R}^2$ and we denote the ϵ -fattening of a subset $S \subset \mathbb{R}^2$ by

$$[S]_\epsilon := \bigcup_{x \in S} B(x, \epsilon).$$

Our aim is to prove that the sequence $(U_n)_{n \geq 0}$ of compact subsets of $[0, 1] \times [0, 1]$ is converging and we provide an elementary description of the limit set \mathcal{L} .

3.2. The (\star) condition. Some pairs of words $(u, v) \in L \times L$ have the property that not only $\binom{u}{v} \equiv 1 \pmod 2$ but also $\binom{uw}{vw} \equiv 1 \pmod 2$ for all words w . Such a property creates a particular pattern occurring in U_n for all $n \geq |u|$.

Definition 12. Let $(u, v) \in L \times L$. We say that (u, v) satisfies the (\star) condition, if $(u, v) \neq (\varepsilon, \varepsilon)$,

$$\binom{u}{v} \equiv 1 \pmod 2, \binom{u}{v0} = 0 \text{ and } \binom{u}{v1} = 0.$$

In particular, this condition implies that $|v| \leq |u|$.

Note that if (u, v) satisfies (\star) , then we clearly have

$$(2) \quad \binom{u}{vw} = 0$$

for all non-empty words w .

Example 13. Some pairs (u, v) satisfying (\star) :

u	1	101	1001	1101	1110
v	1	11	11	111	10

Lemma 14. *If $(u, v) \in L \times L$ satisfies (\star) , then both $(u0, v0)$ and $(u1, v1)$ satisfy the (\star) condition.*

Proof. The fact that $\binom{u0}{v0} \equiv 1 \pmod{2}$ directly follows from Lemma 3. If $\binom{u0}{v00} > 0$ or $\binom{u0}{v01} > 0$ then, $v0$ must appear as a subsequence of u contradicting the assumption. \square

Remark 15. Let (u, v) satisfying (\star) such that $|v| \leq |u| = \ell$. From Remark 11, we know that $(0.0^{\ell-|v|}v, 0.u) + Q/2^\ell \subset U_\ell$. As a consequence of Lemma 14, (u, v) , $(u0, v0)$ and $(u1, v1)$ have an odd binomial coefficient and thus correspond to square regions in $U_{\ell+1}$, i.e.,

$$\{(0.0^{\ell+1-|v|}v, 0.0u), (0.0^{\ell-|v|}v0, 0.u0), (0.0^{\ell-|v|}v1, 0.u1)\} + Q/2^{\ell+1} \subset U_{\ell+1}.$$

Iterating this argument yields, for all $n \geq 0$,

$$\bigcup_{|w|=0}^n (0.0^{\ell+n-|w|-|v|}vw, 0.0^{n-|w|}uw) + Q/2^{\ell+n} \subset U_{\ell+n}.$$

As an example, consider the pair $(101, 11)$ and the corresponding squares in U_3, U_4 and U_5 as depicted in Figure 3.

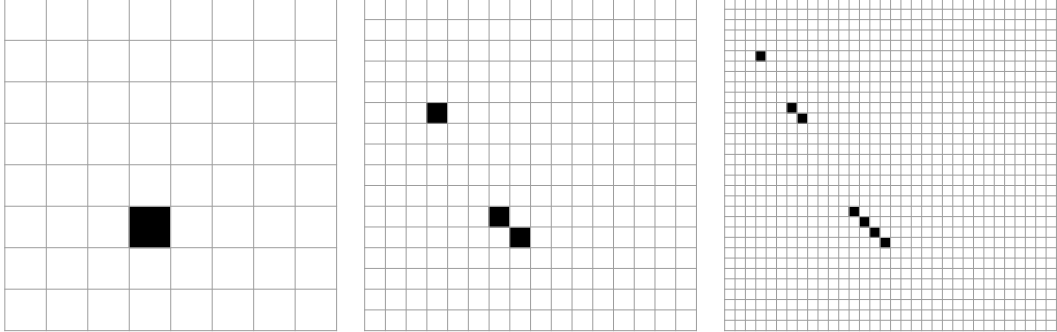


FIGURE 3. The pair $(101, 11)$ satisfying (\star) in U_3, U_4, U_5 .

3.3. Definition of an initial set \mathcal{A}_0 . Observe that, for (u, v) satisfying (\star) , the sequence

$$\left((0.0^{|u|-|v|}v, 0.u) + Q/2^{|u|} \right) \cap U_n \Big|_{n \geq 0}$$

converges to the diagonal of the square $(0.0^{|u|-|v|}v, 0.u) + Q/2^{|u|}$. See for instance, in Figure 3, the square region of size $1/8$ and upper-left corner $(3/8, 5/8)$: in U_3 we have one black square, which is divided into two black squares of size $1/16$ in U_4 and then, four black squares of size $1/32$ in U_5 . This observation leads us to the definition of the set \mathcal{A}_0 .

Definition 16. Let (u, v) in $L \times L$ such that $|u| \geq |v| \geq 1$. We define a closed segment $S_{u,v}$ of slope 1 and length $\sqrt{2} \cdot 2^{-|u|}$ in $[0, 1] \times [1/2, 1]$. The endpoints of $S_{u,v}$ are given by

$$A_{u,v} := (0.0^{|u|-|v|}v, 0.u) \text{ and } B_{u,v} := (0.0^{|u|-|v|}v + 2^{-|u|}, 0.u + 2^{-|u|}).$$

Note that if we allow infinite binary expansions ending with ones, we have

$$B_{u,v} := (0.0^{|u|-|v|}v111\dots, 0.u111\dots).$$

Observe that $S_{u,v}$ is included in $[1/2^{|u|-|v|+1}, 1/2^{|u|-|v|}] \times [1/2, 1]$.

First, let us discuss relative positions of two segments of the form $S_{u,v}$ and in particular, we explain when $S_{s,t} \subset S_{u,v}$.

Lemma 17. *Let (u, v) and (s, t) in $L \times L$ such that $|u| \geq |v| \geq 1$ and $|s| \geq |t| \geq 1$. The point $A_{s,t}$ belongs to $A_{u,v} + [0, 2^{-|u|}] \times [0, 2^{-|u|}]$ if and only if there exist two words w, z of the same length such that $s = uw$ and $t = vz$.*

Proof. It follows from base-2 expansions of integers. Observe that, if $|w| \neq |z|$, then $0.0^{|u|-|v|+|w|-|z|}vz$ does not belong to the interval $[0.0^{|u|-|v|}v, 0.0^{|u|-|v|}v + 2^{-|u|}]$. \square

Corollary 18. *Let (u, v) and (s, t) satisfying (\star) . If the point $A_{s,t}$ belongs to $A_{u,v} + [0, 2^{-|u|}] \times [0, 2^{-|u|}]$, then $S_{s,t} \subset S_{u,v}$.*

Proof. We know that there exist two words w, z of the same length such that $s = uw$ and $t = vz$. From Lemma 4, we get

$$\binom{s}{t} = \binom{uw}{vz} = \sum_{\substack{f,g \in L \\ vz=fg}} \binom{u}{f} \binom{w}{g}.$$

If $|f| > |v|$, then v is a strict prefix of f and thus $\binom{u}{f} = 0$ applying (2). If $|f| < |v|$, then z is a strict suffix of g . In particular, in this case $|w| < |g|$ and thus $\binom{w}{g} = 0$. We conclude that

$$\binom{s}{t} = \binom{u}{v} \binom{w}{z}.$$

Since (s, t) satisfies (\star) , we have $w = z$ and the conclusion follows. \square

Corollary 19. *Let (u, v) and (s, t) satisfying (\star) . Considering the two segments $S_{u,v}$ and $S_{s,t}$, either one is included into the other or, $S_{u,v} \cap S_{s,t}$ is empty or reduced to a common endpoint.*

Definition 20. Let us define the following compact set which is the closure of a countable union of segments

$$\mathcal{A}_0 := \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying}(\star)}} S_{u,v}}.$$

Notice that Definition 16 implies that $\mathcal{A}_0 \subset [0, 1] \times [1/2, 1]$.

Example 21. In Figure 4, each pair (u, v) satisfying (\star) with $|u| \leq 6$ corresponds to a point. We have represented all the corresponding segments $S_{u,v}$. Corollary 19 shows that the set of segments has a partial ordering for inclusion, the maximal segments are the diagonals of the dotted squares. For instance, consider the gray square whose upper-left corner corresponds to $(1101, 111)$ satisfying (\star) . Its diagonal is the segment $S_{1101,111}$ with origin $(13/16, 7/16)$ which has length $\sqrt{2}/16$ and contains the four segments $S_{1101w,111w}$ of length $\sqrt{2}/64$ for $|w| = 2$.

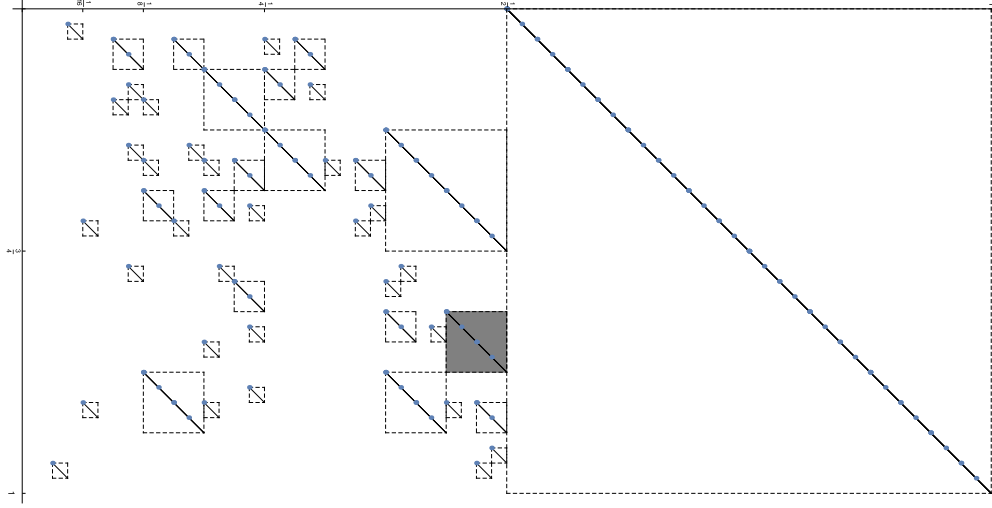


FIGURE 4. An approximation of \mathcal{A}_0 computed with words of length ≤ 6 .

Remark 22. In the definition of \mathcal{A}_0 , we take the closure of a union to ensure the compactness of the set. Here is an example of a limit point that does not belong to the union of segments but to the closure \mathcal{A}_0 . For all $n \geq 0$ and all $r \in \{0, 1, \dots, 7\}$, the pair $(10^{8n+4+r}1, 10^{8n+r}1)$ satisfies (\star) if and only if $0 \leq r \leq 3$. Indeed, observe that

$$\binom{10^{8n+4+r}1}{10^{8n+r}1} = \binom{8n+4+r}{8n+r}.$$

Now, we use Theorem 6 and first assume that $0 \leq r \leq 3$. Let $\text{rep}_2(n) = n_k \cdots n_0$ and $\text{val}_2(r_1 r_0) = r$ with $r_0, r_1 \in \{0, 1\}$, then

$$\binom{8n+4+r}{8n+r} = \binom{n_k}{n_k} \cdots \binom{n_0}{n_0} \binom{1}{0} \binom{r_1}{r_1} \binom{r_0}{r_0} \equiv 1 \pmod{2}.$$

It is easy to see that $(10^{8n+4+r}1, 10^{8n+r}1)$ satisfies the other two conditions of (\star) . If $7 \geq r \geq 4$, when applying Theorem 6, the corresponding product contains a factor $\binom{0}{1}$ and the result is thus even. Consequently, we have a sequence of segments in \mathcal{A}_0 with one endpoint being of the form $(1/2 + 1/2^m, 1/32 + 1/2^m)$ with $m \geq 6$ and the point $(1/2, 1/32)$ is an accumulation point of \mathcal{A}_0 . In Figure 5, we have represented the segments corresponding to $n = 0, 1$ and $r = 0, \dots, 3$.

3.4. Definition of the sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$. In the following definition, we introduce another sequence of compact sets obtained by transforming \mathcal{A}_0 under iterations of two maps. This new sequence, which is shown to be a Cauchy sequence, will allow us to define properly the limit set \mathcal{L} .

Definition 23. We let c denote the homothety of center $(0, 0)$ and ratio $1/2$ and consider the map $h : (x, y) \mapsto (x, 2y)$. Now define a sequence of compact sets

$$\mathcal{A}_n := \bigcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq i}} h^j(c^i(\mathcal{A}_0)).$$

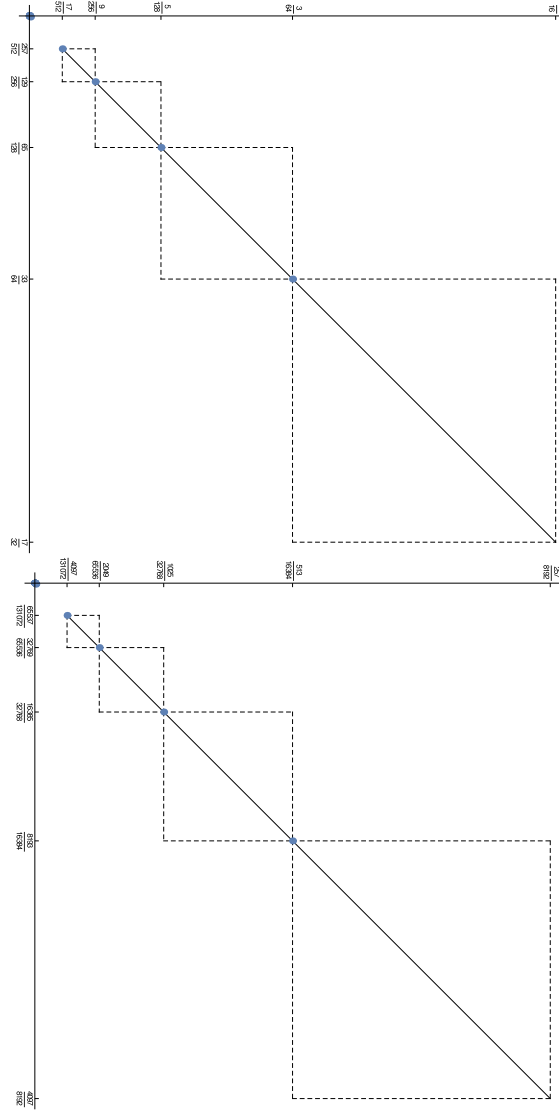
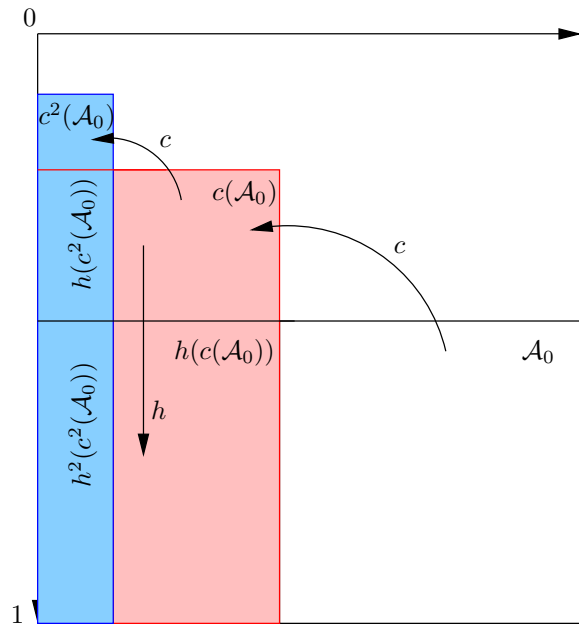
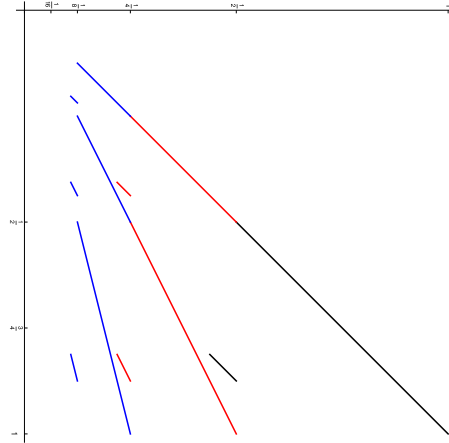


FIGURE 5. A zoom on \mathcal{A}_0 in $[17/2^9, 1/2^4] \times [27/2^9, 17/2^5]$ and in the smaller area $[4097/2^{17}, 257/2^{13}] \times [65537/2^{17}, 4097/2^{13}]$.

Let m, n with $m \leq n$. Using Figure 6, observe that

$$(3) \quad \mathcal{A}_m \cap ([1/2^{m+1}, 1] \times [0, 1]) = \mathcal{A}_n \cap ([1/2^{m+1}, 1] \times [0, 1]).$$

Example 24. In Figure 7, we have depicted two original segments in \mathcal{A}_0 (in black), then one application of c possibly followed by h (in red), then a second application of c followed by at most 2 applications of h (in blue). For instance, the segment $S_{1,1}$ with endpoints $(1/2, 1/2)$ and $(1, 1)$ belongs to \mathcal{A}_0 . Thus, the segment $h^p(c^{p+j}(S_{1,1}))$ with endpoints $(1/2^{p+j+1}, 1/2^{j+1})$ and $(1/2^{p+j}, 1/2^j)$ belongs

FIGURE 6. Two applications of c and h from \mathcal{A}_0 .FIGURE 7. A subset of \mathcal{A}_2 .

to \mathcal{A}_{p+j} for all $p \geq 0$ and all $j \geq 0$. Since $\mathcal{A}_p \subset \dots \subset \mathcal{A}_{2p}$, considering the union of segments of the latter form for $j = 0, \dots, p$, the segment with endpoints $(1/2^{2p+1}, 1/2^{p+1})$ and $(1/2^p, 1)$ belongs to \mathcal{A}_{2p} .

Lemma 25. *The sequence $(\mathcal{A}_n)_{n \geq 0}$ is a Cauchy sequence.*

Proof. Let $\epsilon > 0$. Take N such that $1/2^{N/2} < \epsilon$. Let $n > m > N$. From (3), $[\mathcal{A}_m]_\epsilon$ contains $\mathcal{A}_n \cap ([1/2^{m+1}, 1] \times [0, 1])$. Assume first that m is even. Since \mathcal{A}_m contains the segment with endpoints $(1/2^{m+1}, 1/2^{m/2+1})$ and $(1/2^{m/2}, 1)$ (details are given in the previous example), then $[\mathcal{A}_m]_\epsilon$ contains $[0, 1/2^{m+1}] \times [0, 1]$. If m is odd,

\mathcal{A}_{m-1} and also \mathcal{A}_m contain the segment with endpoints $(1/2^m, 1/2^{(m-1)/2+1})$ and $(1/2^{(m-1)/2}, 1)$ and the same conclusion follows. Thus $[\mathcal{A}_m]_\epsilon$ contains \mathcal{A}_n . Since $\mathcal{A}_m \subset \mathcal{A}_n$, we directly have that $[\mathcal{A}_n]_\epsilon$ contains \mathcal{A}_m . \square

Definition 26. Since we have a Cauchy sequence in the complete metric space $(\mathcal{H}(\mathbb{R}^2), d_h)$, the limit of $(\mathcal{A}_n)_{n \geq 0}$ is a well defined compact set denoted by \mathcal{L} . An approximation of this set is given in Figure 9 in the appendix.

4. CONVERGENCE OF $(U_n)_{n \geq 0}$ TO \mathcal{L}

Now we will show that the sequence $(U_n)_{n \geq 0}$ of compact subsets of $[0, 1] \times [0, 1]$ converges to \mathcal{L} . The first part is to show that, when ϵ is a positive real number, then $U_n \subset [\mathcal{L}]_\epsilon$ for all sufficiently large n .

Lemma 27. *Let $\epsilon > 0$. For all $n \in \mathbb{N}$ such that $2^{-n+1} < \epsilon$, we have*

$$U_n \subset [\mathcal{L}]_\epsilon.$$

Proof. Let $\epsilon > 0$. Choose n such that $2^{-n+1} < \epsilon$.

Let $(x, y) \in U_n$. From Remark 11, there exists $(u, v) \in L \times L$ such that $\binom{u}{v} \equiv 1 \pmod 2$, $|u| \leq n$ and

$$(x, y) \in (0.0^{n-|v|}v, 0.0^{n-|u|}u) + Q/2^n.$$

Assume first that (u, v) satisfies (\star) . The segment $S_{u,v}$ of length $\sqrt{2} \cdot 2^{-|u|}$ having $A_{u,v} = (0.0^{|u|-|v|}v, 0.u)$ as endpoint belongs to \mathcal{A}_0 . Now apply $n - |u|$ times the homothety c to this segment. So the segment $c^{n-|u|}(S_{u,v})$ of length $\sqrt{2} \cdot 2^{-n}$ of endpoint $(0.0^{n-|v|}v, 0.0^{n-|u|}u)$ belongs to $\mathcal{A}_{n-|u|}$ and thus to \mathcal{L} . Hence $d((x, y), \mathcal{L}) < \epsilon$.

Now assume that (u, v) does not satisfy (\star) . By assumption, we have an odd number r of occurrences of v in u . For each occurrence of v in u , we count the total number of zeroes after it. We thus define a sequence of non-negative integer indices

$$|u| \geq i_1 \geq i_2 \geq \dots \geq i_r \geq 0$$

corresponding to the number of zeroes following the first, the second, ..., the r th occurrence of v in u . Now let k be a non-negative integer such that $k > \lceil \log_2 |u| \rceil$. We get

$$\binom{u0^{2^k}1}{v0^{2^k}1} = \sum_{\ell=1}^r \binom{2^k + i_\ell}{2^k}.$$

Indeed, for each $\ell \in \{1, \dots, r\}$, consider the ℓ th occurrence of v in u : we have the factorization $u = pw$ where the last letter of p is the last letter of the ℓ th occurrence of v and $|w|_0 = i_\ell$. With this particular occurrence of v , we obtain occurrences of $v0^{2^k}1$ in $u0^{2^k}1$ by choosing 2^k zeroes among the $2^k + i_\ell$ zeroes available in $w0^{2^k}1$. Moreover, with the long block of 2^k zeroes, it is not possible to have any other occurrence of $v0^{2^k}1$ than those obtained from occurrences of v in u .

For each $\ell \in \{1, \dots, r\}$, we have

$$\binom{2^k + i_\ell}{2^k} \equiv 1 \pmod 2$$

from Theorem 6. Since r is odd, we get

$$\binom{u0^{2^k}1}{v0^{2^k}1} \equiv 1 \pmod 2.$$

Now, for all $k \in \mathbb{N}$ such that $k > \lceil \log_2 |u| \rceil$, it is easy to check that $(u0^{2^k}1, v0^{2^k}1)$ satisfies (\star) . For the sake of clarity, if k is a non-negative integer such that $k > \lceil \log_2 |u| \rceil$, we define

$$u_k := u0^{2^k}1 \quad \text{and} \quad v_k := v0^{2^k}1.$$

As in the first part of the proof, the segment S_{u_k, v_k} of length $\sqrt{2} \cdot 2^{-|u|-2^k-1}$ having $A_{u_k, v_k} = (0.0^{|u|-|v|}v0^{2^k}1, 0.u0^{2^k}1)$ as endpoint belongs to \mathcal{A}_0 . Now apply $n - |u|$ times the homothety c to this segment. So the segment $c^{n-|u|}(S_{u_k, v_k})$ of length $\sqrt{2} \cdot 2^{-n-2^k-1}$ of endpoint $(0.0^{n-|v|}v0^{2^k}1, 0.0^{n-|u|}u0^{2^k}1)$ belongs to $\mathcal{A}_{n-|u|}$ and thus to \mathcal{L} . Hence $d((x, y), \mathcal{L}) < \epsilon$. \square

Let $\epsilon > 0$. It remains to show that, for all sufficiently large $n \in \mathbb{N}$, $\mathcal{L} \subset [U_n]_\epsilon$.

Lemma 28. *Let $\epsilon > 0$. For all $(x, y) \in \mathcal{L}$, there exists N such that for all $n \geq N$, $d((x, y), U_n) < \epsilon$.*

Proof. Let $\epsilon > 0$ and let $(x, y) \in \mathcal{L}$. Since $(\mathcal{A}_n)_{n \geq 0}$ converges to \mathcal{L} , there exists N_1 and $(x', y') \in \mathcal{A}_{N_1}$ such that,

$$d((x, y), (x', y')) < \epsilon/4.$$

By definition of \mathcal{A}_{N_1} , there exist i, j such that $0 \leq j \leq i \leq N_1$ and $(x'_0, y'_0) \in \mathcal{A}_0$ such that

$$h^j(c^i((x'_0, y'_0))) = (x', y').$$

By definition of \mathcal{A}_0 , there exists a pair $(u, v) \in L \times L$ satisfying (\star) and $(x''_0, y''_0) \in S_{u, v}$ such that

$$d((x'_0, y'_0), (x''_0, y''_0)) < \epsilon/4.$$

Notice that, since $j \leq i$,

$$\begin{aligned} d((x', y'), h^j(c^i((x''_0, y''_0)))) &= d(h^j(c^i((x'_0, y'_0))), h^j(c^i((x''_0, y''_0)))) \\ &\leq d((x'_0, y'_0), (x''_0, y''_0)) < \epsilon/4. \end{aligned}$$

Consequently, we get that

$$d((x, y), h^j(c^i((x''_0, y''_0)))) < \epsilon/2.$$

In the second part of the proof, we will show that $d(h^j(c^i((x''_0, y''_0))), U_n) < \epsilon/2$ for all sufficiently large n . We will make use of the constants i, j and words u, v given above.

Let $n \geq 0$. Since $(u, v) \in L \times L$ satisfies (\star) , the pairs (uw, vw) satisfy (\star) for all words w of length $n, \dots, n+i$. In particular, if w is a word of length n , since $\binom{uw}{vw} \equiv 1 \pmod{2}$, applying Lemma 5, at least one of the two binomial coefficients $\binom{uw0}{vw}$, $\binom{uw1}{vw}$ is odd. Iterating this argument j times, we conclude that at least one of the 2^j binomial coefficients of the form $\binom{uwz}{vw}$ with $|z| = j$ is odd. Otherwise stated, at least one of the square regions

$$(0.0^{i+|u|-|v|}vw, 0.0^{i-j}uwz) + Q/2^{n+i+|u|}, \quad \text{with } |z| = j,$$

is a subset of $U_{n+i+|u|}$. We observe that each of these square regions is intersected by $h^j(c^i(S_{u, v}))$. Indeed, the latter segment has $(0.0^{i+|u|-|v|}v, 0.0^{i-j}u)$ and $(0.0^{i+|u|-|v|}v111\dots, 0.0^{i-j}u111\dots)$ as endpoints and slope 2^j . This can be visualized in Figure 8 where each rectangular gray region contains at least one square region from $U_{n+i+|u|}$. Consequently, every point¹ of $h^j(c^i(S_{u, v}))$ is at distance at most

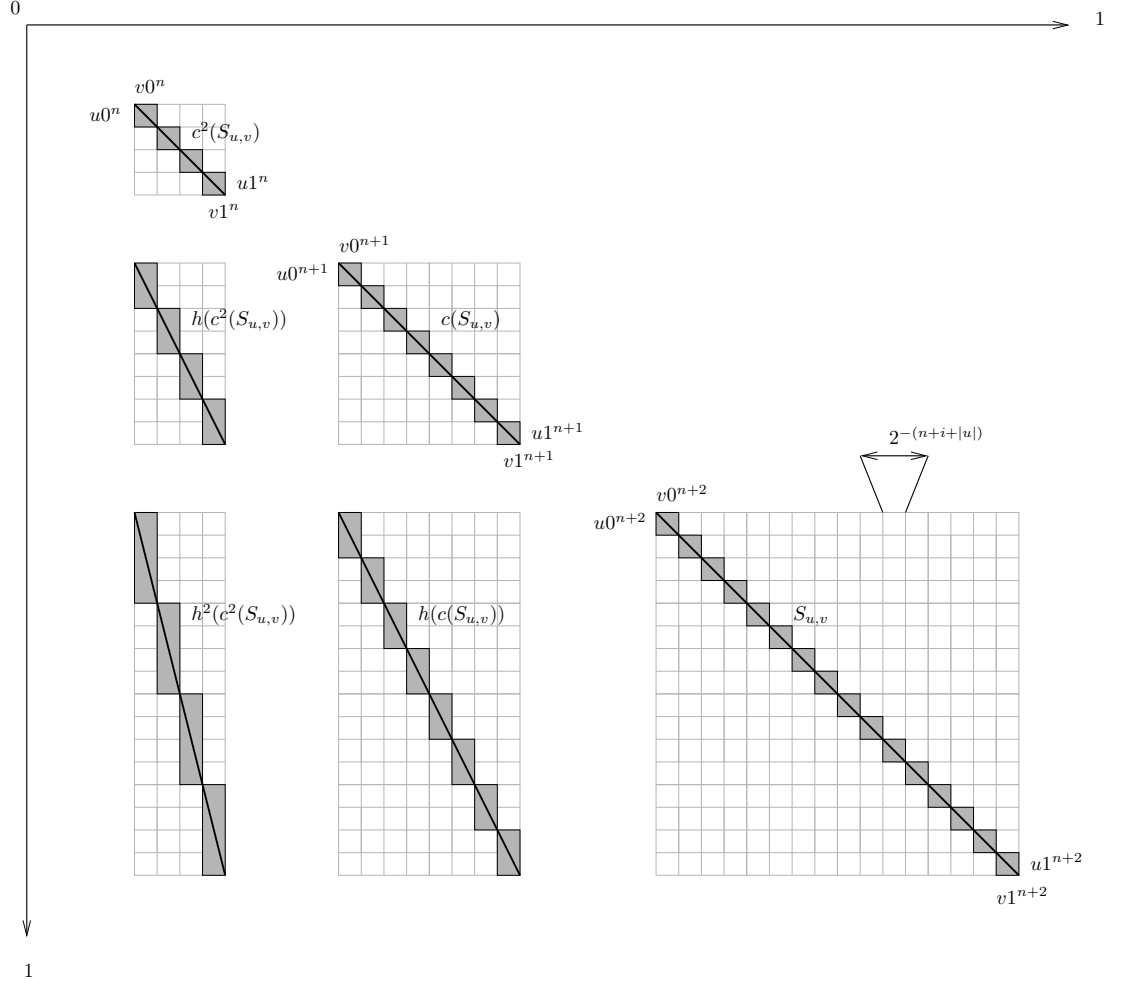


FIGURE 8. Situation occurring in the proof of Lemma 28.

$2^j/2^{n+i+|u|}$ from a point in $U_{n+i+|u|}$. In particular, this holds for $h^j(c^i((x''_0, y''_0)))$. We choose N_2 such that $2^j/2^{N_2+i+|u|} < \epsilon/2$. Hence, for all $n \geq N_2 + i + |u|$,

$$d(h^j(c^i((x''_0, y''_0))), U_n) < \epsilon/2.$$

To conclude the proof, for all $n \geq N_2 + i + |u|$, we have

$$d((x, y), U_n) < \epsilon.$$

□

Theorem 29. *The sequence $(U_n)_{n \geq 0}$ converges to \mathcal{L} .*

¹For every point of $h^j(c^i(S_{u,v}))$, this observation will permit us to build a sequence $((f_n, g_n))_{n \geq 0}$ converging to it and such that $(f_n, g_n) \in U_n$, for all n , and the distance between two consecutive elements of the sequence tends to zero when n tends to infinity.

Proof. Let $\epsilon > 0$. From Lemma 27, it suffices to show that $\mathcal{L} \subset [U_n]_\epsilon$ for all sufficiently large $n \in \mathbb{N}$. For all $(x, y) \in \mathcal{L}$, from the proof of Lemma 28, there exists a sequence $((f_i(x, y), g_i(x, y)))_{i \geq 0}$ such that $(f_i(x, y), g_i(x, y)) \in U_i$, for all i , and there exists $N_{(x, y)}$ such that, for all $i, j \geq N_{(x, y)}$,

$$(4) \quad d((f_i(x, y), g_i(x, y)), (f_j(x, y), g_j(x, y))) < \epsilon/2$$

and

$$d((f_i(x, y), g_i(x, y)), (x, y)) < \epsilon/2.$$

We trivially have

$$\mathcal{L} \subset \bigcup_{(x, y) \in \mathcal{L}} B((f_{N_{(x, y)}}(x, y), g_{N_{(x, y)}}(x, y)), \epsilon/2).$$

Since \mathcal{L} is compact, we can extract a finite covering: there exist $(x_1, y_1), \dots, (x_k, y_k)$ in \mathcal{L} such that

$$\mathcal{L} \subset \bigcup_{j=1}^k B((f_{N_{(x_j, y_j)}}(x_j, y_j), g_{N_{(x_j, y_j)}}(x_j, y_j)), \epsilon/2).$$

Let $N = \max_{j=1, \dots, k} N_{(x_j, y_j)}$. From (4), we deduce that, for all $j \in \{1, \dots, k\}$ and all $n \geq N$,

$$B((f_{N_{(x_j, y_j)}}(x_j, y_j), g_{N_{(x_j, y_j)}}(x_j, y_j)), \epsilon/2) \subset B((f_n(x_j, y_j), g_n(x_j, y_j)), \epsilon)$$

and therefore

$$\mathcal{L} \subset \bigcup_{j=1}^k B((f_n(x_j, y_j), g_n(x_j, y_j)), \epsilon) \subset [U_n]_\epsilon.$$

□

5. EXTENSION MODULO p

In this paper, for the sake of simplicity, we have only considered odd binomial coefficients. It is straightforward to adapt our reasonings, constructions and results to a more general setting. Let p be a fixed prime and $r \in \{1, \dots, p-1\}$. We can extend Definition 7 to

$$T_{n, r} := \bigcup \left\{ (\text{val}_2(v), \text{val}_2(u)) + Q \mid u, v \in L_n, \binom{u}{v} \equiv r \pmod{p} \right\} \subset [0, 2^n] \times [0, 2^n]$$

and introduce corresponding sets $U_{n, r}$ as in Definition 10. Since we make use of Lucas' theorem, we limit ourselves to congruences modulo a prime. We just sketch the main differences with the case $p = 2$. See, for instance, Figure 11 for the case $p = 3$ and $r = 2$.

In this setting, analogously to Lemma 5, one can observe that if $\binom{u}{vb} \equiv r \pmod{p}$ for some $b \in \{0, 1\}$ then there exists $a \in \{0, 1\}$ such that $\binom{ua}{vb} \equiv r \pmod{p}$. This observation is useful to adapt the proof of Lemma 28.

The (\star) condition becomes $(\star)_r$

$$\binom{u}{v} \equiv r \pmod{p}, \quad \binom{u}{v0} = 0 \text{ and } \binom{u}{v1} = 0$$

and Lemma 14 still holds. Note that the pairs (u, v) satisfying this condition depend on the choice of p and r . The sets \mathcal{A}_n are defined as before.

Remark 30. The pair $(1, 1)$ satisfies $(\star)_r$ if and only if $r = 1$. Thus, the segment with endpoints $(1/2^{2m+1}, 1/2^{m+1})$ and $(1/2^m, 1)$ belongs to \mathcal{A}_{2m} only if $r = 1$. This observation made in Example 24 was used in the proof of Lemma 25.

We present an alternative proof of Lemma 25 in this general setting.

Proof of Lemma 25 (generalization). Let $\epsilon > 0$. Choose N such that $2^{-N} < \epsilon$. Let m, n such that $n > m > N$. From (3), $[\mathcal{A}_m]_\epsilon$ contains $\mathcal{A}_n \cap ([1/2^{m+1}, 1] \times [0, 1])$. It remains to show that $[\mathcal{A}_m]_\epsilon$ contains $\mathcal{A}_n \cap [0, 1/2^{m+1}] \times [0, 1]$. The pairs $(u0^{|u|}1^r0, u0^{|u|}10)$ satisfy $(\star)_r$ for all words $u \in 1\{0, 1\}^*$. In this case, the segment of endpoints $(0.0^{r-1}u0^{|u|}10, 0.u0^{|u|}1^r0)$ and $(0.0^{r-1}u0^{|u|}10 + 2^{2|u|+r+1}, 0.u0^{|u|}1^r0 + 2^{2|u|+r+1})$ has length $\sqrt{2} 2^{-(2|u|+r+1)}$ and belongs to \mathcal{A}_0 . If we apply m times c and $j \in \{0, \dots, m\}$ times h to the latter segment, we obtain a segment $S(u, j)$ with endpoint

$$A(u, j) := (0.0^{m+r-1}u0^{|u|}10, 0.0^{m-j}u0^{|u|}1^r0)$$

and of length less than $\sqrt{2} 2^{-(2|u|+r+1)}$ that belongs to \mathcal{A}_m . Note that this segment $S(u, j)$ lies into $R(j) := [1/2^{m+r}, 1/2^{m+r-1}] \times [1/2^{m-j+1}, 1/2^{m-j}]$. Observe that

$$\bigcup_{u \in 1\{0, 1\}^*} A(u, j)$$

is the diagonal of the region $R(j)$. Consequently, for all $j \leq m$, we have

$$\bigcup_{u \in 1\{0, 1\}^*} [S(u, j)]_\epsilon \supset [0, 1/2^{m+1}] \times [1/2^{m-j+1}, 1/2^{m-j}]$$

since $2^{-(m+r-1)} \leq 2^{-N} < \epsilon$. Letting j range from 0 to m , we deduce that $[\mathcal{A}_m]_\epsilon$ contains $[0, 1/2^{m+1}] \times [0, 1]$ and therefore $\mathcal{A}_n \cap ([0, 1/2^{m+1}] \times [0, 1])$. Since $\mathcal{A}_m \subset \mathcal{A}_n$, we directly have that $[\mathcal{A}_n]_\epsilon$ contains \mathcal{A}_m . \square

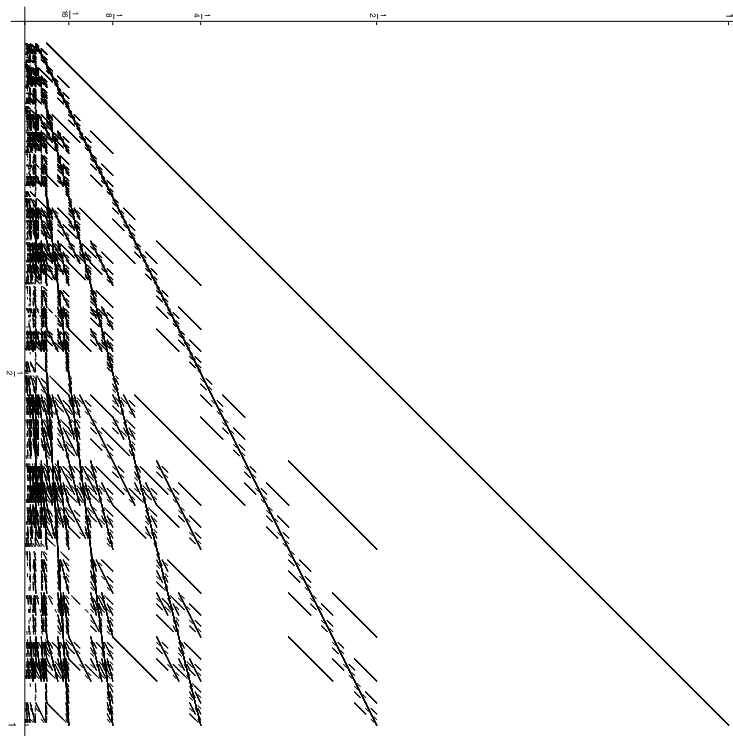
The proof of Lemma 27 follows the same lines. Simply replace the word $u0^{2^k}1$ (resp. $v0^{2^k}1$) with $u0^{p^k}1$ (resp. $v0^{p^k}1$). We may apply Lucas' theorem with base- p expansions.

6. APPENDIX

Example 31. We have represented in Figure 9 the 1370 segments of \mathcal{A}_0 for words of length at most 8 (we are missing segments of length $\leq \sqrt{2}/2^9$) and we have applied the maps $h^j(c^i(\cdot))$ to this set of segments for $0 \leq j \leq i \leq 4$. Thus we have an approximation of \mathcal{A}_4 . Excepting the segments of length $\leq \sqrt{2}/2^9$ and their images, we have an exact image of \mathcal{L} inside $[1/32, 1] \times [0, 1]$.

Example 32. We have represented the set U_9 in Figure 10.

Example 33. We have represented in Figure 11 the set $U_{7,2}$ when considering binomial coefficients congruent to 2 modulo 3 and an approximation of the limit set \mathcal{L} proceeding as in Example 31.

FIGURE 9. An approximation of the limit set \mathcal{L} .

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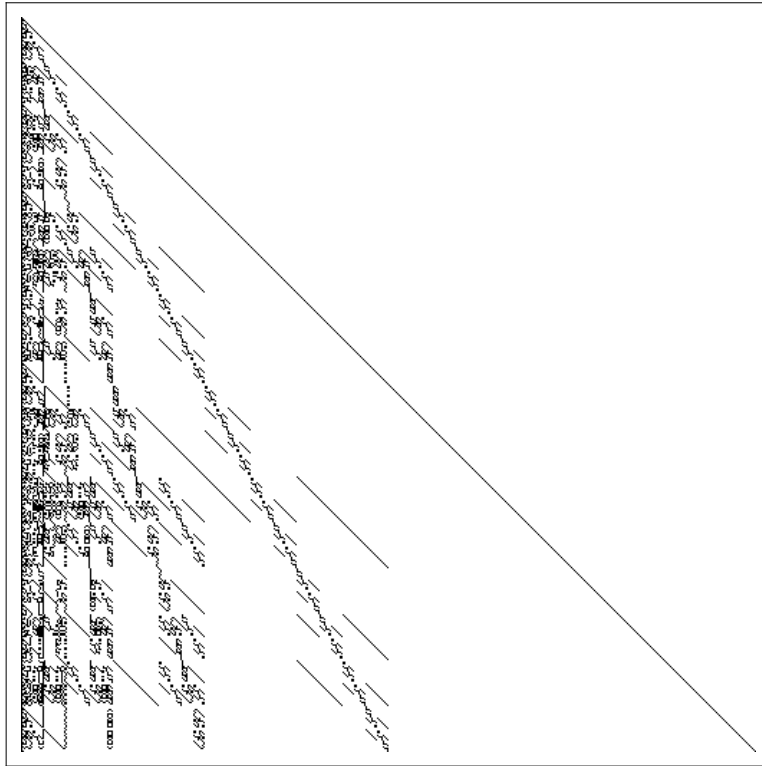


FIGURE 10. The set U_9 .

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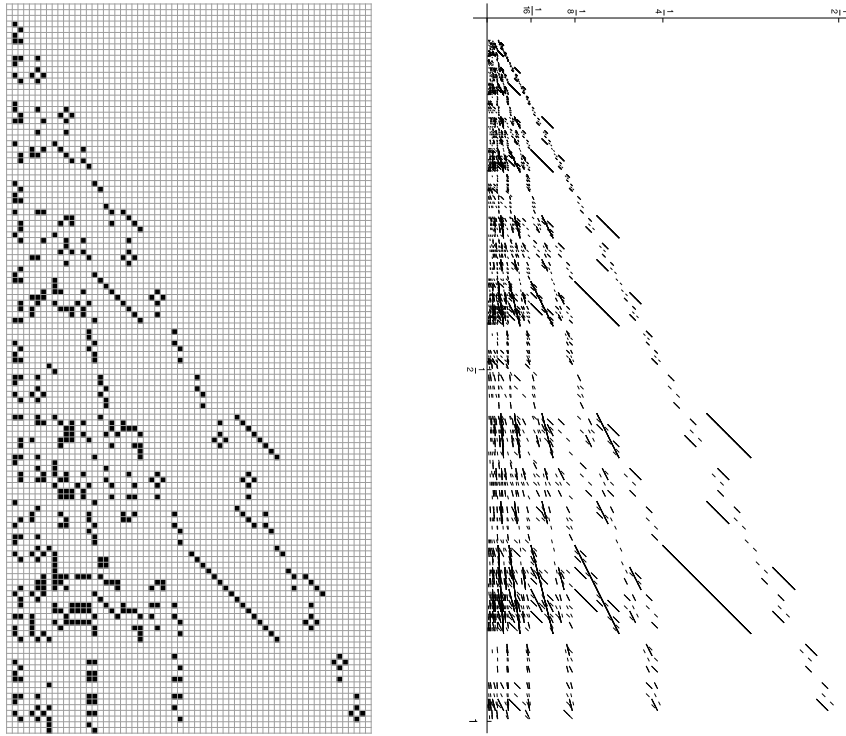


FIGURE 11. The set $U_{7,2}$ and an approximation of the corresponding set \mathcal{L} .