TOPOLOGY OPTIMIZATION OF MECHANICAL AND AEROSPACE COMPONENTS SUBJECT TO FATIGUE STRESS CONSTRAINTS

Pierre DUYSINX*, Maxime COLLET*, Simon BAUDUIN*, Emmanuel TROMME*, Lise NOEL*, and Matteo BRUGGI+

* Aerospace and Mechanical Engineering Dept, University of Liege, Belgium
+ Dept of Civil and Environmental Engineering, Politecnico di Milano, Italy

10th World Congress on Structural and Multidisciplinary Optimization
WCSMO-11, Sydney, June 8 to 12, 2015

LAY-OUT

- Introduction
- Topology optimization subject to stress constraints
  - Problem statement
- Fatigue criterion
  - Sines, Crossland
  - Goodman
- Sensitivity Analysis
- Numerical Application
- Conclusion and Perspectives

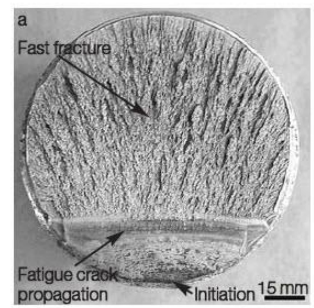
INTRODUCTION

INTRODUCTION

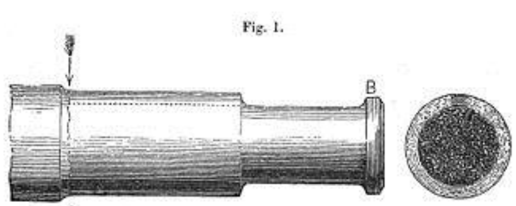
- For metallic materials, failure can happen at a much lower load level compared to the static application if the loading is the result of a cyclic application

- In mechanical and aerospace engineering, fatigue is responsible for 80% of the structural failures

Versailles train accident, 1842



Typical fatigue failure



INTRODUCTION

- To reduce the risk of failure, one can oversize the structure but increasing the weight is detrimental for:
  - Human manipulation
  - Fuel consumption
  - Cost of product…

- Engineering design has to find the best compromise between weight and risk of failure

- Replacing slow and inefficient trial-and-error approaches, one can resort to Topology Optimization to suggest new design concepts

TOPOLOGY OPTIMIZATION PROBLEM

- Optimal material distribution within a given domain

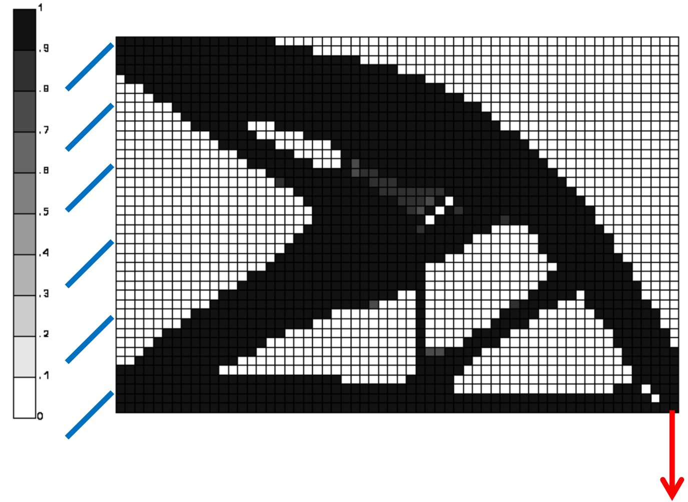
- Discretization of displacements and density distribution using FEM

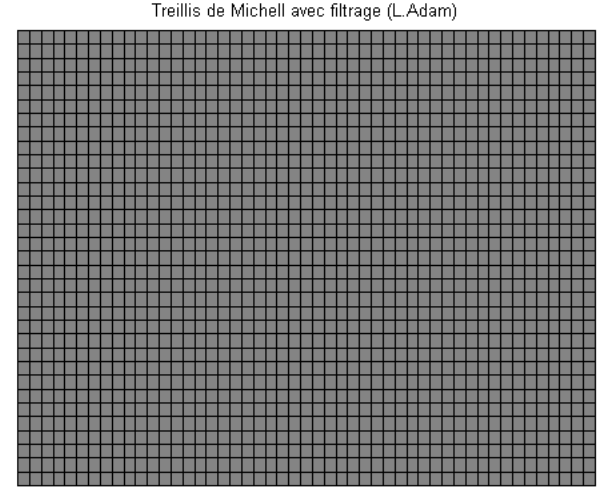
$$KU = F$$

- Interpolation of material properties between void and solid and penalize intermediate densities (SIMP model)

$$E_j(x_j) = E_{min} + x_j^p \, (E_0 - E_{min})$$

- Solve optimization problem using efficient MP optimizers with continuous variables (e.g. MMA)



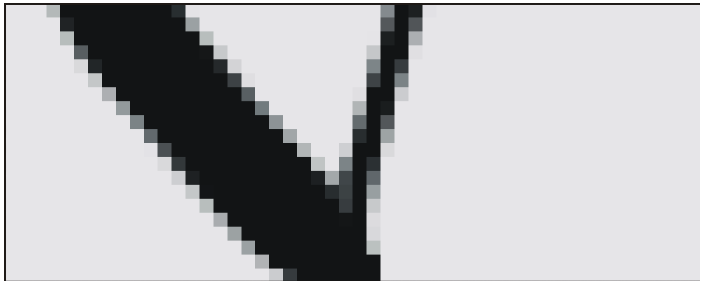


TOPOLOGY OPTIMIZATION PROBLEM

- Compliance design
  - Usual approach
  - Unable to capture the specific character of stress constraints

$$\min_{0<\boldsymbol{x}\leq 1} \quad \boldsymbol{F}^T\boldsymbol{U}$$

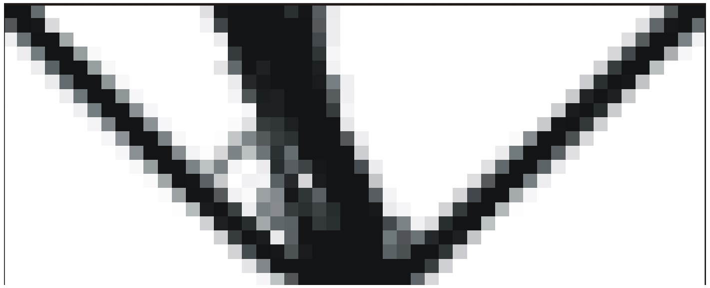
$$\text{s.t.} \quad V = \sum_{e=1} v_e x_e \leq \bar{V}$$



- Stress constrained design
  - Technical difficulties to be solved
  - Define appropriate failure criterion ➔ extension to fatigue!

$$\min_{0<\boldsymbol{x}\leq 1} \quad \max_{e} ||\sigma_e(\boldsymbol{x})||$$

$$\text{s.t.} \quad V = \sum_{e=1} v_e x_e \leq \bar{V}$$



TOPOLOGY OPTIMIZATION

- Challenges of of stress constraints in topology optimization
  - Definition of relevant stress criteria at microscopic level
    - Microscopic stress should be considered

$$\sigma_{ij}^{M} = E_{ijkl}\,\epsilon_{kl}^{M} \quad \Rightarrow \quad <\sigma_{ij}^{e}>= \frac{\sigma_{ij}^{e,M}}{x_e^q}$$

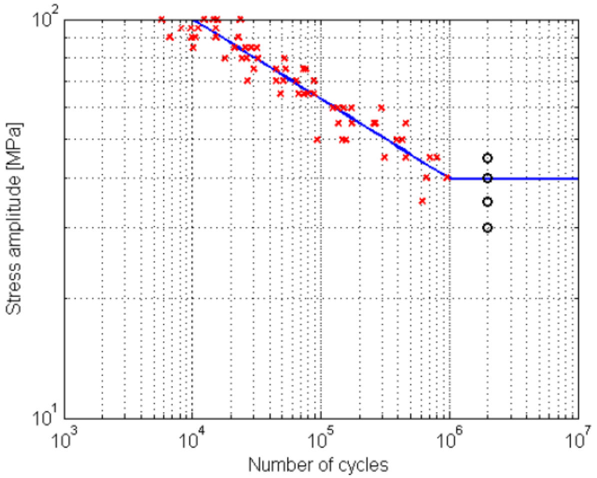
  - Stress singularity phenomenon:
    - ε-relaxation (Chang and Guo, 1992)
    - q-p relaxation (Bruggi, 2008)

$$<\sigma_{ij}^{e}> = \frac{x_e^p}{x_e^q} E_{ijkl}^{0}\varepsilon_{kl}^{0} = x_e^{p-q} E_{ijkl}^{0}\varepsilon_{kl}^{0} \qquad q < p \quad \text{and} \quad q \nearrow p$$

  - Large scale optimization problem
    - Local constraints
    - Aggregation of constraints: P-norm $\left[\sum_{e=1}^{N} (<||\sigma_e||>)^P\right]^{1/P}$

FATIGUE (UNI AXILAL CASE)

- Wöhler's curve : fundamental work
  - Reduction of the amplitude of stress with the number of cycles

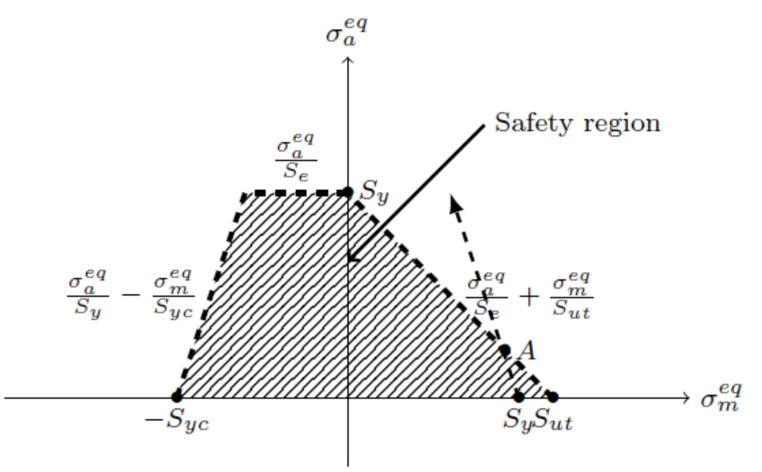


- Goodman diagram:
  - Influence of mean and alternate stress components
  - Line of equal failure probability for a certain number of cycles

$$\sigma(t) = \sigma_m + \sigma_a \sin(\omega t)$$

  - Amplitude / mean stress

$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2} \qquad \sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$$



MULTI AXIAL FATIGUE CRITERIA

- Design against fatigue: some measure, the effective stress, of the stress tensor may never exceed some critical material dependent value.

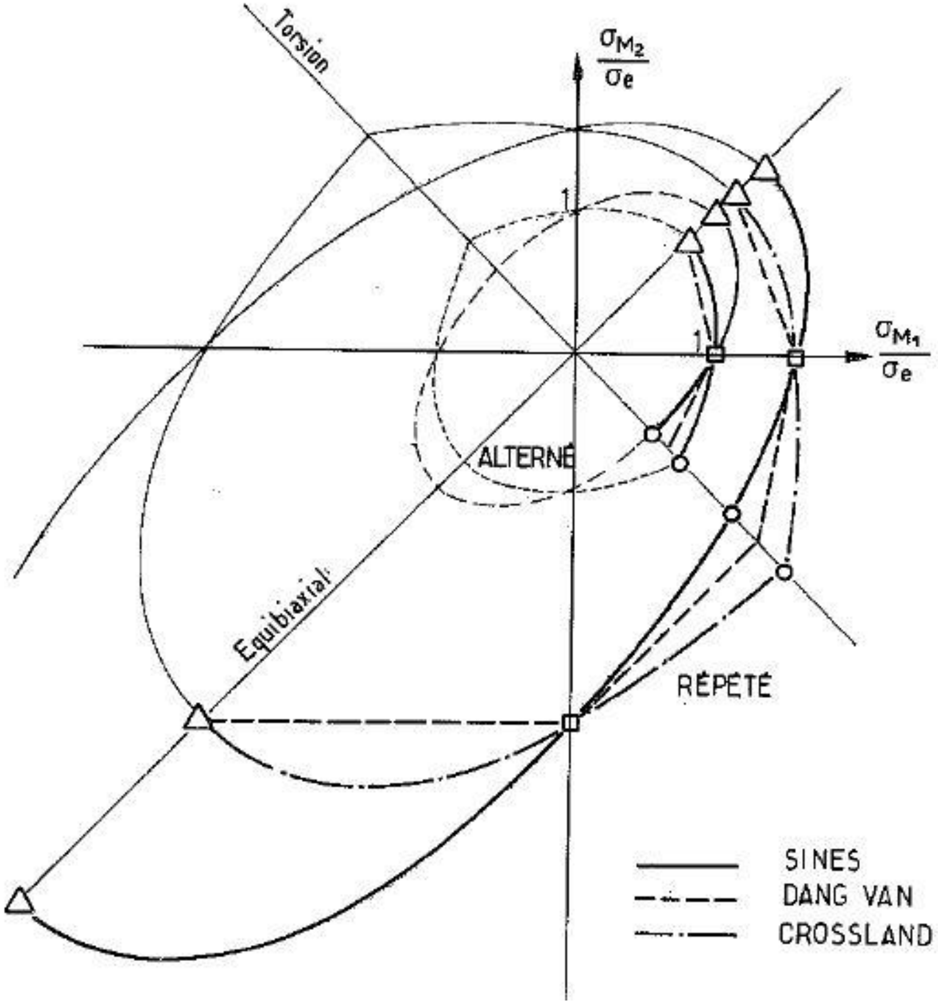
- Local models: fatigue strength depends only on the local value of the effective stress:
  - Sines, Crossland…

$$||\sigma|| = \sigma_{VM}(\sigma_a) + k\,\sigma_h(\sigma_m)$$

  - Matake, Dang-Van, Findley: the fatigue resistance is ruled by the stress acting on the specific plane exhibiting the worst fatigue loading
    - Stress vector acting on the plane of normal n $\boldsymbol{T}_n = \sigma\,\boldsymbol{n}$
    - The effective stress to consider

$$||\sigma|| = \max_{n}\ \mathrm{f}(\boldsymbol{T}_n(\boldsymbol{n}))$$

MULTI AXIAL FATIGUE CRITERIA: CROSSLAND



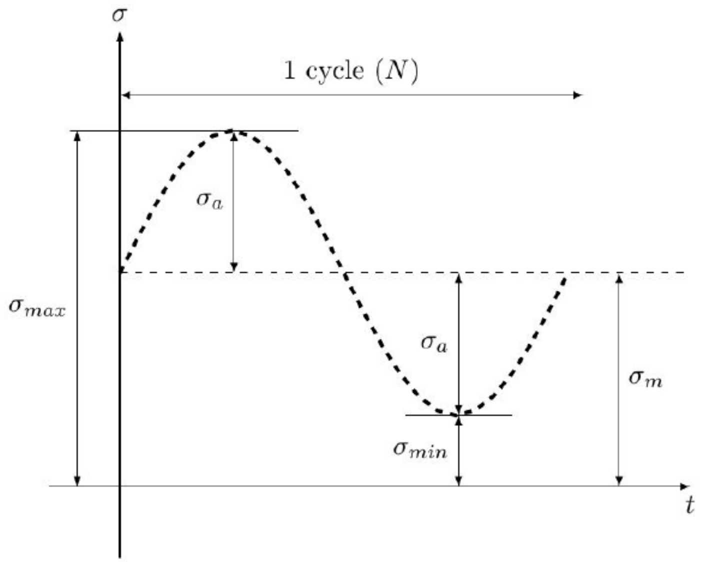
MULTI AXIAL FATIGUE CRITERIA

- Like in 1-D problem let's assume that the total stress is given by a certain amount of alternate component $c_a\ \sigma_a$ and a given amount of mean component $c_m\ \sigma_m$:

$$\sigma_{tot} = c_a \sigma_a + c_m \sigma_m$$

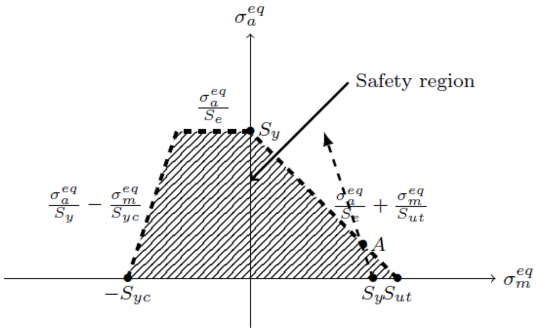
$$0 \leq c_a, c_m \leq 1$$

$$c_a + c_m = 1.$$



- In the following, let assume that alternate and mean components are defined by the same reference load case.

MULTI AXIAL FATIGUE CRITERIA: SINES



- Sines fatigue criterion reads

$$\sqrt{J_{2,a}} + \kappa \cdot \sigma_{h,m} \leq \lambda$$

  - Where

$$\lambda = t_{-1} \qquad \kappa = \frac{6t_{-1}}{f_0} - \sqrt{6}$$

  - With t_{-1}, the fatigue limit in reverse torsion and f_0 is the fatigue in repeated bending

- For plane stress

$$J_{2,a} = \frac{1}{6}\left[(\sigma_{11,a} - \sigma_{22,a})^2 + \sigma_{22,a}^2 + \sigma_{11,a}^2 + 6\sigma_{12,a}^2\right]$$

$$\sigma_{h,m} = \frac{1}{3}(\sigma_{11,m} + \sigma_{22,m}) = \frac{J_1}{3}$$

MULTI AXIAL FATIGUE CRITERIA: SINES

- Reminding also that

$$\sigma_a^{eq} = \sqrt{3J_2(\sigma_{a,ij})} \qquad \sigma_m^{eq} = J_1(\sigma_{m,ij})$$

- Sines criterion can be restated in term of the first and second stress invariants

$$\frac{\sigma_a^{eq}}{\sqrt{3}\lambda} + \kappa\frac{\sigma_m^{eq}}{3\lambda} \leq 1$$

- Remarks:
  - Similar expression to Prager Drucker and Ishai criteria considered for unequal stress constraints
  - Alternate and mean components are computed from the same reference load case, each one accounting for the fraction c_a and c_m of the reference load case

MULTI AXIAL FATIGUE CRITERIA: SINES

- Assuming a SIMP model, after Finite Element discretization, one can calculate the stresses at appropriate positions (e.g. the element centroïd) using the tension matrix $\mathbf{T}_e^0$

$$\sigma_{ij} = x^p E^0_{ijkl} \varepsilon_{kl} \quad \Rightarrow \quad \boxed{\sigma_e = x_e^p \mathbf{T}_e^0 \mathbf{U}}$$

- First and second invariants can be computed by introducing the hydrostatic stress matrix $\mathbf{H}_e^0$ and the von Mises quadratic stress matrix $\mathbf{M}_e^0$:

$$\begin{aligned} J_{1,e}(\sigma_{ij}) &= x_e^p \mathbf{H}_e^0 \mathbf{U}_e \\ 3 J_{2D,e}(\sigma_{ij}) &= x_e^{2p} \mathbf{U}_e^T \mathbf{M}_e^0 \mathbf{U}_e \end{aligned}$$

- It is easy to recover the value of the alternate and mean stress components

$$\begin{aligned} \sigma_{a,e}^{eq} &= x_e^p \left(c_a \sqrt{\mathbf{U}_\mathbf{e}^\mathbf{T} \mathbf{M}_\mathbf{e}^0 \mathbf{U}_\mathbf{e}} \right) = x_e^p \overline{\sigma}_{a,e}^{eq} \\ \sigma_{m,e}^{eq} &= x_e^p (c_m \mathbf{H}_\mathbf{e}^0 \mathbf{U}_\mathbf{e}) = x_e^p \overline{\sigma}_{m,e}^{eq} \end{aligned}$$

MULTI AXIAL FATIGUE CRITERIA: SINES

- For topology optimization, as suggested by Duysinx & Bendsoe (1998), one should consider the micro stresses after applying the polarization factor

$$\langle \sigma_{ij,e} \rangle = \frac{\sigma_{ij,e}}{x_e^q}$$

- Sines criterion for topology optimization writes

$$\frac{\langle \sigma_{a,e}^{eq} \rangle}{\sqrt{3}\lambda} + \kappa \frac{\langle \sigma_{m,e}^{eq} \rangle}{3\lambda} \leq 1$$

- The final expression Sines criterion for topology optimization reads

$$\frac{x_e^{(p-q)}}{\lambda} \left[\frac{\overline{\sigma}_{a,e}^{eq}}{\sqrt{3}} + \frac{\kappa}{3} \overline{\sigma}_{m,e}^{eq} \right] \leq 1$$

MULTI AXIAL FATIGUE CRITERIA: CROSSLAND

- Crossland fatigue criterion is very similar to Sines citerion

$$\sqrt{J_{2,a}} + \kappa \cdot \sigma_{h,max} \leq \lambda$$

- Difference lies in the fact in Crossland the hydrostatic term is evaluated on the basis of the maximum stress (not only on the mean component): $\sigma_{max} = \sigma_a + \sigma_m$:

$$\sigma_{h,max} = \sigma_{h,a} + \sigma_{h,m}$$

- The criterion writes

$$\frac{\sigma_a^{eq}}{\sqrt{3}\lambda} + \frac{\kappa \sigma_M^{eq}}{3\lambda}$$

MULTI AXIAL FATIGUE CRITERIA: CROSSLAND

- Evaluating the quantities using a finite element method, one has

$$\left\{ \begin{array}{lcccc} \sigma_{a,e}^{eq} & = & x_e^p \left(c_a \sqrt{\mathbf{U_e^T M_e^0 U_e}} \right) & = & x_e^p \overline{\sigma}_{a,e}^{eq} \\ \sigma_{M,e}^{eq} & = & x_e^p (c_a \mathbf{H_e^0 U_e} + c_m \mathbf{H_e^0 U_e}) & = & x_e^p \overline{\sigma}_{M,e}^{eq} \end{array} \right.$$

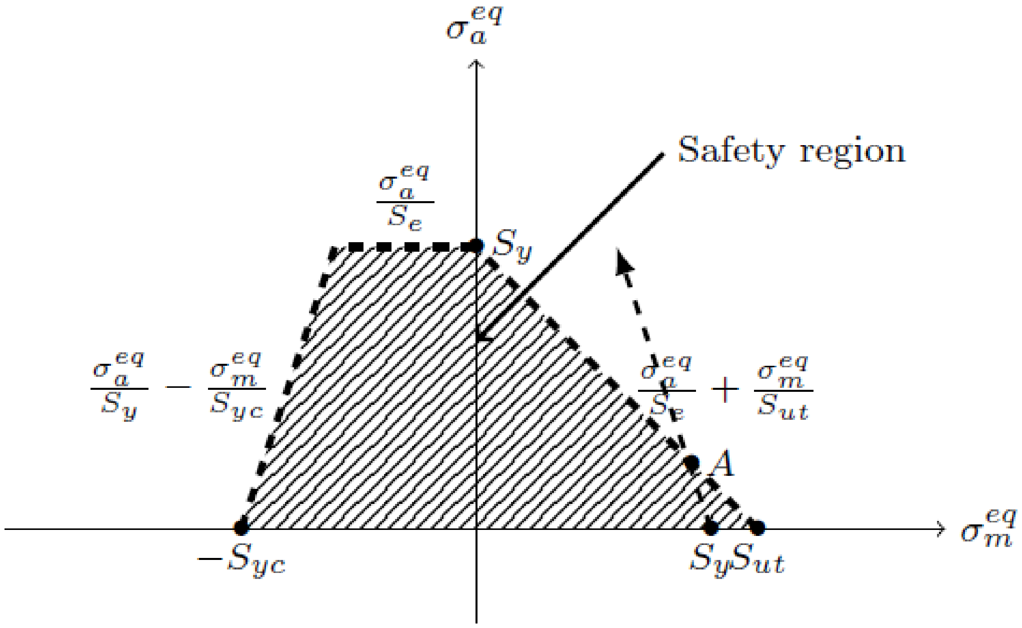
- Within the topology optimization framework, it comes

$$\frac{x_e^{(p-q)}}{\lambda} \left[\frac{\overline{\sigma}_{a,e}^{eq}}{\sqrt{3}} + \frac{\kappa}{3} \overline{\sigma}_{M,e}^{eq} \right] \leq 1$$

FATIGUE: GOODMAN APPROACH

- Goodman diagram: Influence of mean and alternate components

$$\begin{cases} \dfrac{\sigma_a^{eq}}{S_e} + \dfrac{\sigma_m^{eq}}{S_{ut}} \leq 1 \\ \dfrac{\sigma_a^{eq}}{S_y} - \dfrac{\sigma_m^{eq}}{S_{yc}} \leq 1 \\ \dfrac{\sigma_a^{eq}}{S_e} \leq 1 \end{cases}$$



- S_y: yield stress in tension, S_{yc}: yield stress in compression, S_e: fatigue stress (infinite life) and S_{ut}: ultimate stress.

PROBLEM FORMULATION: SINES & CROSSLAND

- Minimum volume with fatigue stress constraints

$$
\begin{cases}
\min\limits_{x_0 \leq x_e \leq 1} & \mathcal{W} = \sum\limits_{N} x_e V_e \\
\text{s.t.} & \mathbf{K}(\mathbf{x})\,\mathbf{U} \;=\; \mathbf{F}, \\
 & \mathcal{C} \,/\, \mathcal{C}_L \;\leq\; 1, \\
 & \dfrac{x_e^{(p-q)}}{\lambda} \left[\dfrac{\overline{\sigma}_{a,e}^{eq}}{\sqrt{3}} + \dfrac{\kappa}{3} \overline{\sigma}_{M,e}^{eq} \right] \leq 1, \\
 & \qquad \text{for } \; e = 1, \ldots, N
\end{cases}
$$

- Compliance constraints is introduced to provide a better stability and effectiveness to the convergence (Bruggi & Duysinx, 2012)

$$\mathcal{C}_L \;=\; \alpha_c \mathcal{C}_0$$

PROBLEM FORMULATION: GOODMAN APPROACH

- Problem formulation for Goodman criterion

$$\begin{cases} \min\limits_{x_0 \le x_e \le 1} \mathcal{W} = \sum\limits_{N} x_e V_e \\ \text{s.t.} \quad \mathbf{K}(\mathbf{x})\ \mathbf{U} \ = \ \mathbf{F}, \\ \mathcal{C} \ / \ \mathcal{C}_L \ \le \ 1, \\ x_e^{(p-q)} \left(\dfrac{\overline{\sigma}_{a,e}^{eq}}{S_e} + \dfrac{\overline{\sigma}_{m,e}^{eq}}{S_{ut}} \right) \le 1, \\ x_e^{(p-q)} \left(\dfrac{\overline{\sigma}_{a,e}^{eq}}{S_y} - \dfrac{\overline{\sigma}_{m,e}^{eq}}{S_{yc}} \right) \le 1, \\ x_e^{(p-q)} \dfrac{\overline{\sigma}_{a,e}^{eq}}{S_e} \le 1, \end{cases}$$

SENSITIVITY ANALYSIS

- Sensitivity analysis of compliance

$$\frac{\partial \mathcal{C}}{\partial x_k} = -p x_k^{p-1} \mathbf{U}_k^T \mathbf{K}_k^0 \mathbf{U}_k,$$

- Sensitivity analysis of fatigue stress criteria requires the sensitivity analysis of the alternate, mean, and max components.
- Deriving the expression of the criteria, it comes

$$\begin{aligned}
\frac{\partial \langle \sigma_{a,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek}(p-q) x_e^{p-q-1}\ \overline{\sigma}_{a,e}^{eq} + \frac{\partial \overline{\sigma}_{a,e}^{eq}}{\partial x_k} x_e^{p-q} \\
\frac{\partial \langle \sigma_{m,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek}(p-q) x_e^{p-q-1}\ \overline{\sigma}_{m,e}^{eq} + \frac{\partial \overline{\sigma}_{m,e}^{eq}}{\partial x_k} x_e^{p-q} \\
\frac{\partial \langle \sigma_{M,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek}(p-q) x_e^{p-q-1}\ \overline{\sigma}_{M,e}^{eq} + \frac{\partial \overline{\sigma}_{M,e}^{eq}}{\partial x_k} x_e^{p-q}.
\end{aligned}$$

SENSITIVITY ANALYSIS

- Selecting the adjoin methods since we have less active stress constraints that the number of design variables, one has:

$$\frac{\partial \overline{\sigma}_{a,e}^{eq}}{\partial x_k} = -\widetilde{\boldsymbol{U}}^T \frac{\partial \boldsymbol{K}}{\partial x_k} \boldsymbol{U} \quad \text{with} \quad \boldsymbol{K}\widetilde{\boldsymbol{U}} = \left[c_a (\boldsymbol{U}^T \boldsymbol{M}_e^0 \boldsymbol{U})^{-\frac{1}{2}} \boldsymbol{M}_e^0 \boldsymbol{U} \right]$$

$$\frac{\partial \overline{\sigma}_{m,e}^{eq}}{\partial x_k} = -\widetilde{\boldsymbol{U}}^T \frac{\partial \boldsymbol{K}}{\partial x_k} \boldsymbol{U}, \quad \text{with} \quad \boldsymbol{K}\widetilde{\boldsymbol{U}} = \left[c_m \boldsymbol{H}_e^0 \right]$$

$$\frac{\partial \overline{\sigma}_{M,e}^{eq}}{\partial x_k} = -\widetilde{\boldsymbol{U}}^T \frac{\partial \boldsymbol{K}}{\partial x_k} \boldsymbol{U}, \quad \text{with} \quad \boldsymbol{K}\widetilde{\boldsymbol{U}} = \left[c_a \boldsymbol{H}_e^0 + c_m \boldsymbol{H}_e^0 \right]$$

NUMERICAL APPLICATION

- Implementation : Topology optimization tool in MATLAB based 88-line code by Andreassen et al. (2011)

- Density filter:

$$\tilde{x}_e = \frac{1}{\sum_N H_{ej}} \sum_N H_{ej} x_j,$$

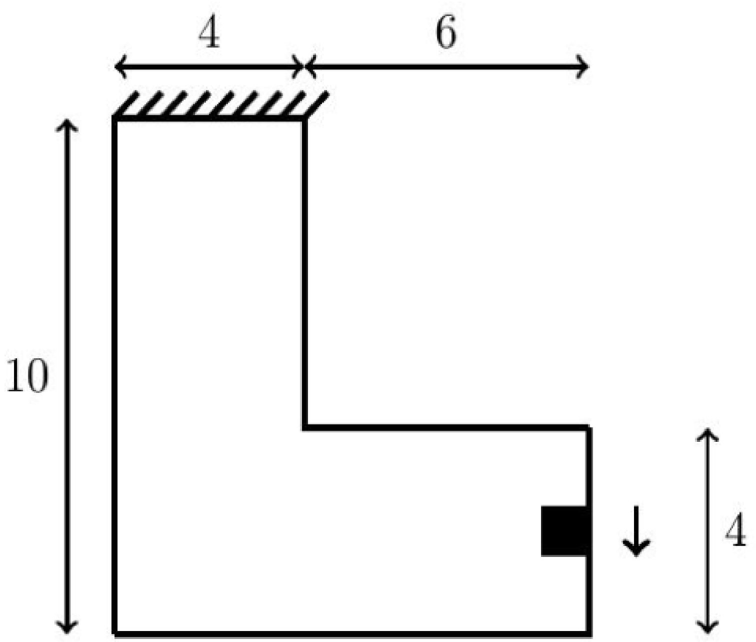
$$H_{ej} = \sum_N \max(0, r_{min} - \mathrm{dist}(e, j)),$$

- MMA solver by Svanberg (1987)

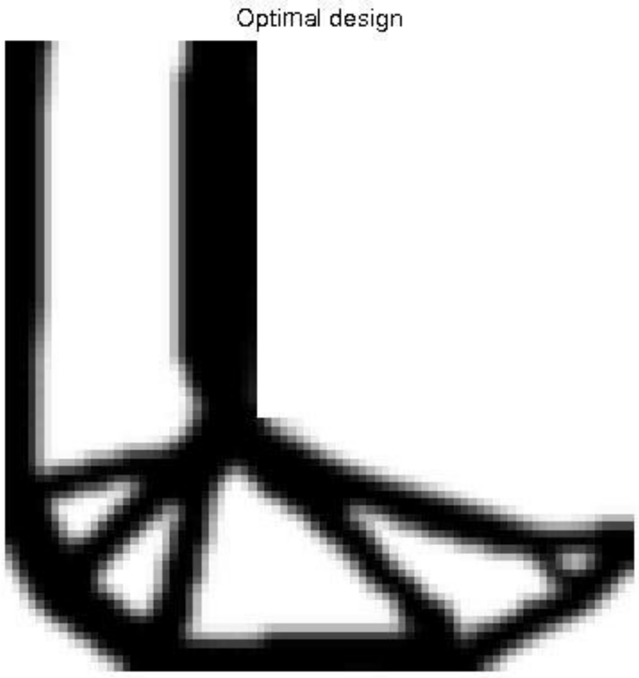
$$\begin{array}{lll}
\min & f_0(\mathbf{x}) + z + \sum_{j=1}^{m} (c_j y_j + \frac{1}{2} d_j y_j^2) & \\
\text{s.t.:} & f_j(\mathbf{x}) - a_j z - y_j \leq 0 & j = 1 \ldots m \\
 & \underline{x}_i \ \leq \ x_i \ \leq \overline{x}_i & i = 1 \ldots n \\
 & y_j \geq 0 & j = 1 \ldots m \\
 & z \geq 0 &
\end{array}$$

NUMERICAL APPLICATION: L-SHAPE

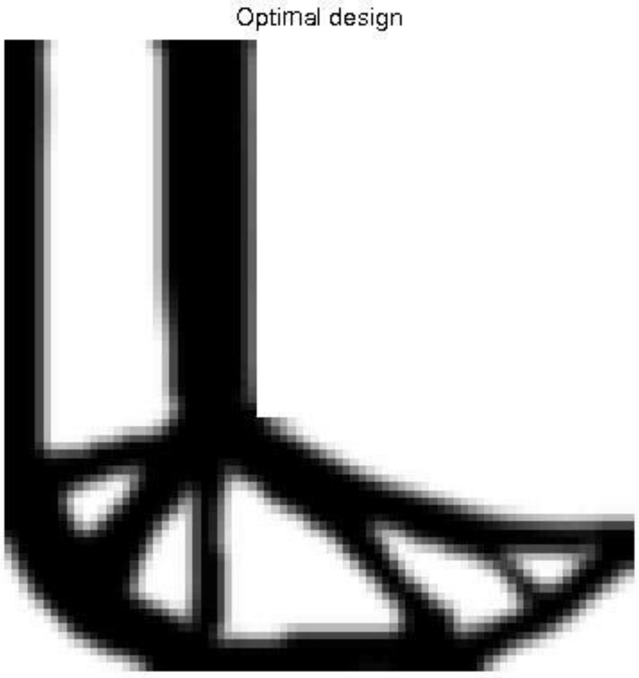
- SIMP model
  - Penalization p=3
  - q-p relaxation: q=2.6 → 2.75
- Load F=95 N
  - c_a = 0.7 and c_m = 0.3
- Material : Steel with properties from Norton (2000)
  - E = 1 Mpa (normalized), ν=0.3
  - σ_f = 580 MPa, t_{-1}= 160 MPa, f_{-1}= 260 MPa
- Compliance regularization constraint: α_c=2



NUMERICAL APPLICATION: LSHAPE

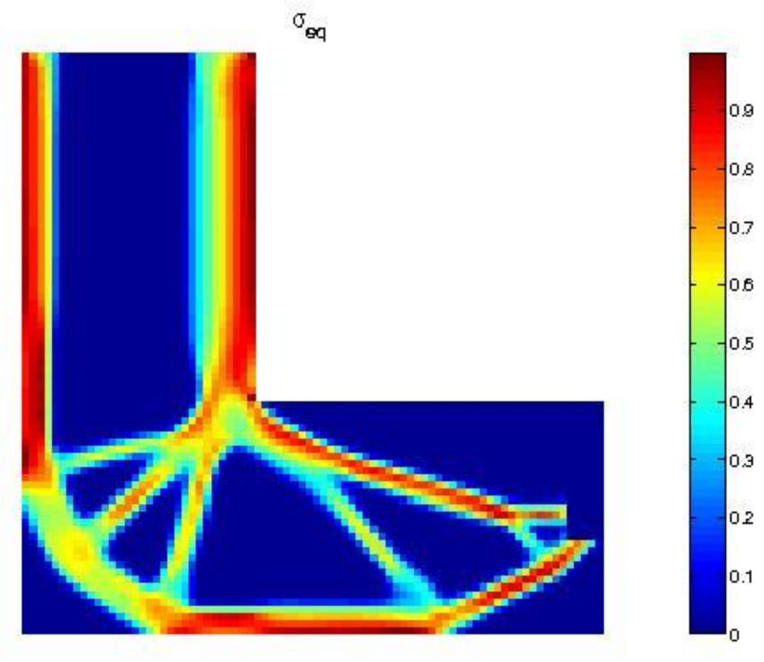


Optimal design with Sines criterion

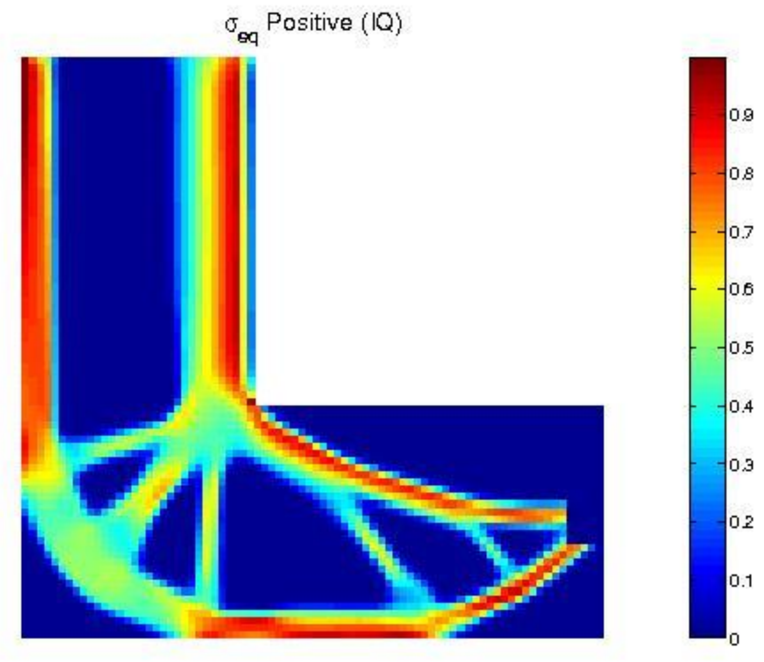


Optimal design with Crossland criterion

NUMERICAL APPLICATION: LSHAPE



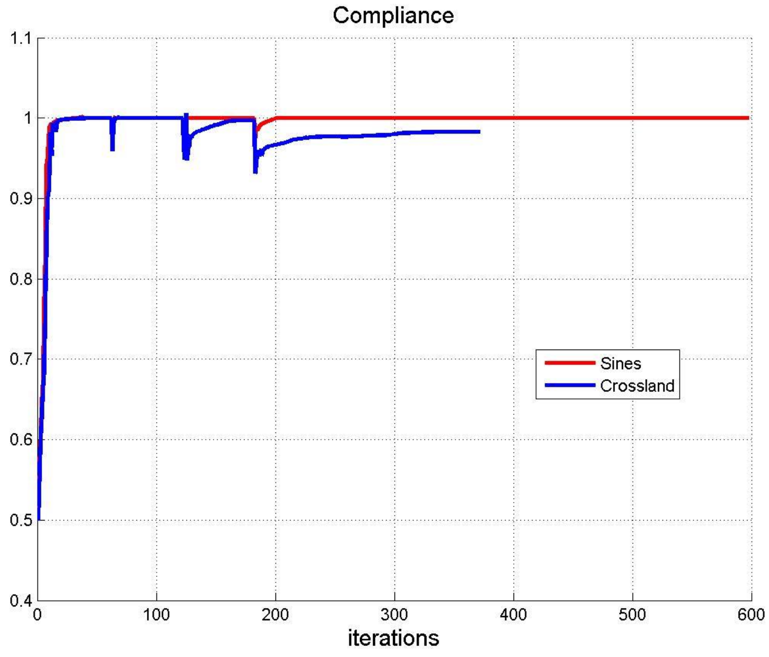
Stress map for
optimal design with Sines criterion



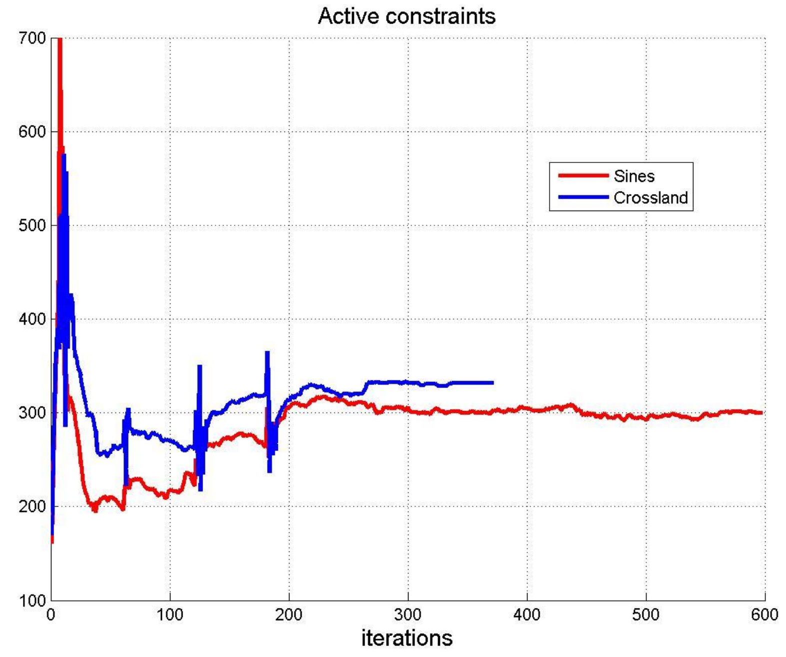
Stress map for
optimal design with Crossland criterion

NUMERICAL APPLICATION: LSHAPE

Problem	N	$\mathcal{W}/\mathcal{W}_0$	$\mathcal{C}/\mathcal{C}_0$	N_a^f
MWS	4096	0.4553	2	299
MWC	4096	0.4991	1.97	332



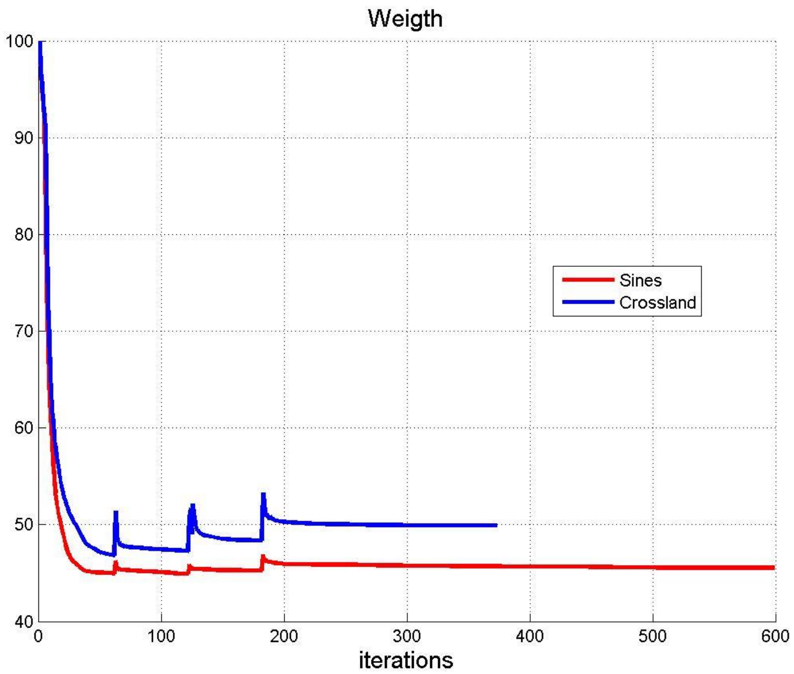
Evolution of the global compliance constraint



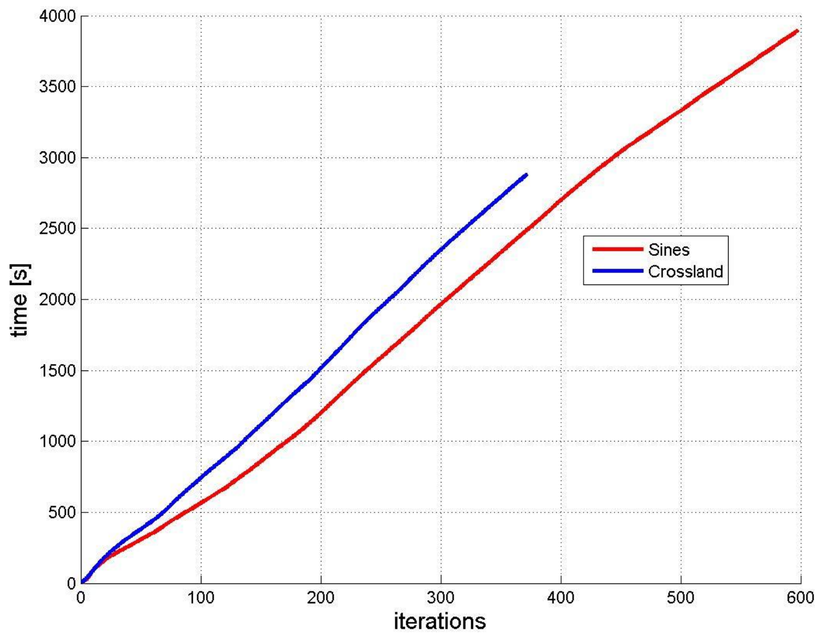
Evolution of the number of active constraints

NUMERICAL APPLICATION: LSHAPE

Problem	N	$\mathcal{W}/\mathcal{W}_0$	$\mathcal{C}/\mathcal{C}_0$	N_a^f
MWS	4096	0.4553	2	299
MWC	4096	0.4991	1.97	332

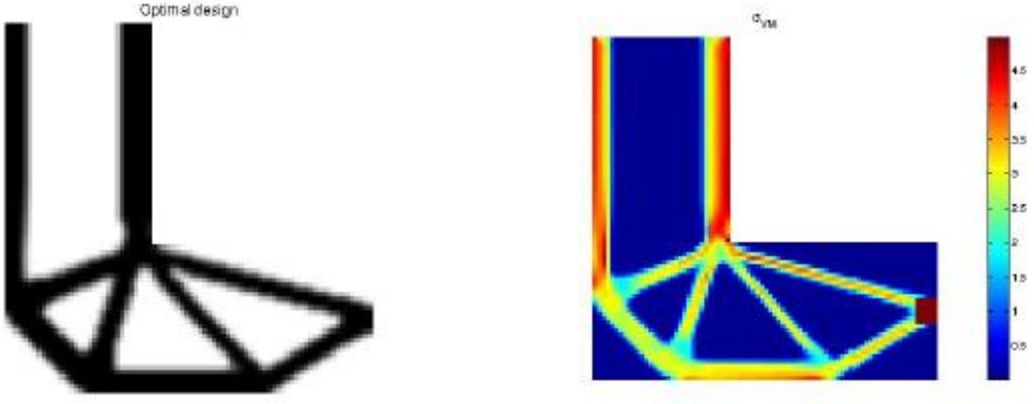


Evolution of the objective function volume



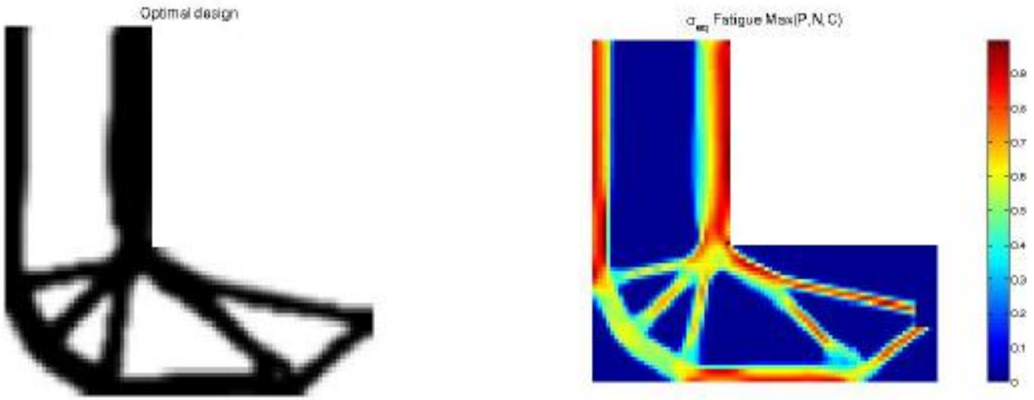
Evolution of the cumulative CPU time

FATIGUE: GOODMAN APPROACH



(a)

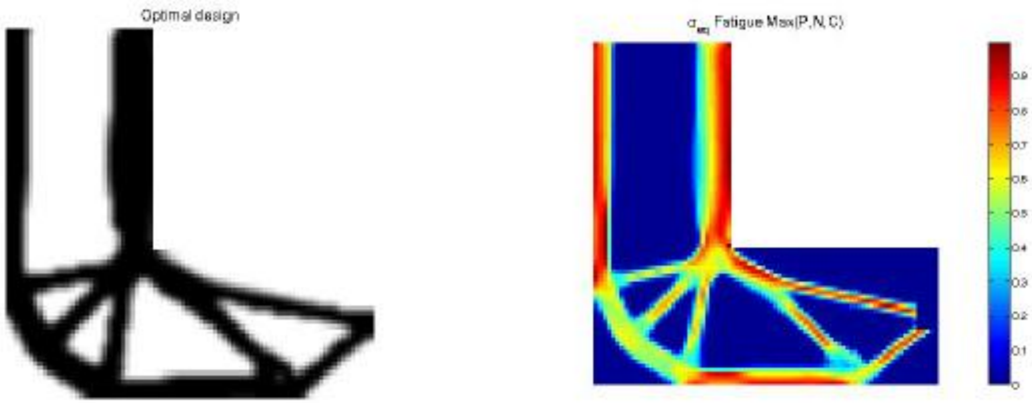
Min volume
s.t. compliance constraint



(b)

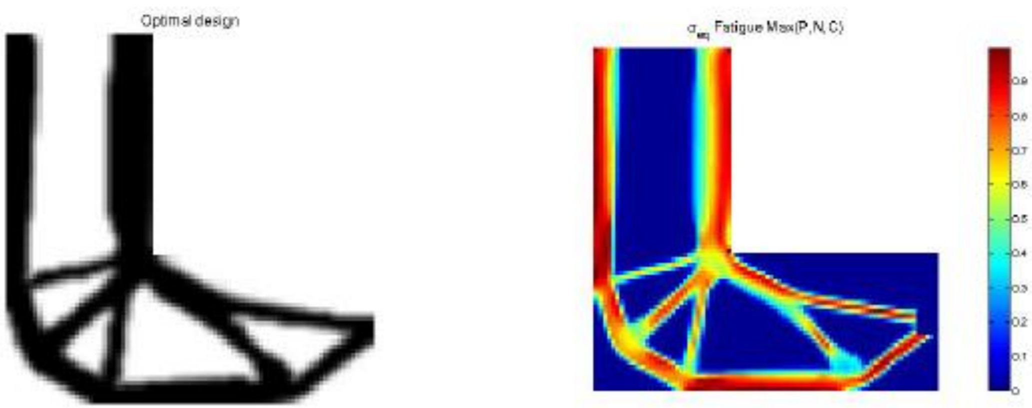
Min volume
s.t. Goodman stress constraint

FATIGUE: GOODMAN APPROACH



(b)

Min volume
s.t. Goodman stress constraint
(same max stress in tension
and compression)



(c)

Min volume
s.t. Goodman stress constraint
(lower max stress in compression
than tension)

CONCLUSIONS
& PERSPECTIVES

CONCLUSIONS

- (First) investigation of fatigue stress criteria that can be used in topology optimization

- Sines and Crossland are classic fatigue criteria:
  - Introduces a dependence in J_1 (hydrostatic pressure) and in J_2 (distortional energy – von Mises) stress invariants like in unequal stress failure criteria
  - Sines and Crossland are similar to Dang Van for a single reference load case
  - Are naturally compliant to be integrated in stress constrained topology optimization
  - Sensitivity analysis can be carried out using
  - Crossland is more restrictive and leads to heavier designs after topology optimization

PERSPECTIVES

- Practical applications calls for further developments extending the method to :
  - Consider stress history $\sigma_i(t)$ instead of a single load case:
    - ➔ other criteria like Matake, Dang Van, Finley…
  - Consider cumulative damage Palmer Milgren

- Increase the efficiency of the solution of the optimization problem

- Consider additive manufacturing constraints

THANK YOU FOR YOUR ATTENTION

THANK YOU FOR
YOUR ATTENTION