Université de Liège

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ANALYSIS OF SOME FINITE DIFFERENCE SCHEMES FOR SLIGHTLY NOISY TIME DEPENDENT SIGNALS

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TABLE OF CONTENT

1 INTRODUCTION

In many scientific applications, the question arises to determine the rate of a time-dependent phenomenon. This is theoretically done by calculating the time derivative of the signal that describes this phenomenon.

However, in practical applications, the analytical form of this signal is not known. Only is available a discrete-time form of this signal, resulting from real-life concrete measurements whose data acquisition process provides a sampled signal.
Various numerical differentiation methods have been developed to compute an approximated derivative of the signal from its discrete values. Amongst them, one of the most commonly applied is the finite difference method.

As an example, to the knowledge of the authors, all the European fire testing laboratories perform some deflection measurements with dedicated sensors (position transducers designed for the direct, absolute measurement of displacement). Each laboratory then computes the rate of deflection from the deflection measures by its own numerical methods. Some essential test results are directly associated with these calculated rates.

The characteristics of the sampling and the choice of the numerical methods may influence the outcome, among which:

- the sampling period of the data acquisition,
- the numerical differentiation method for the calculation of the rate,
- any additional numerical low-pass filtering.

The present study will focus on the moving-average filter on one hand, and on differentiation by some usual finite difference schemes on the other hand. Amongst relevant parameters, the influence of the sampling period and the differentiation step will be examined, in particular regarding their consequence on the time-accuracy and the noise filtering effect of the considered methods.

The study will focus in detail on the first differentiation methods. The results for the first order backward second differentiation method will also be provided without much comment. Anyway, the theory developed here for the first differentiation methods applies identically to any kind of finite difference scheme.

1.1 NOTATIONS

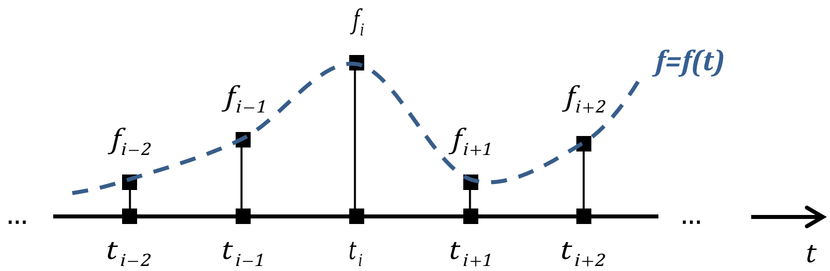


Figure 1 – Signal sampling notation

- $f = f(t)$: continuous time-dependant signal, only known through its measured values at times t_i
- $f^{(n)} = \dfrac{d^n f}{dt^n}$: n^th time-derivative of f

- $t_i = i\Delta t$: sampled time (i is an integer)
- $\Delta t = t_i - t_{i-1}$: sample step, or time step (sampling period)
- $f_i = f(t_i)$: sampled signal, i.e. signal measured at time t_i
- $\tilde{f}_i$: sampled filtered signal
- $f_i^{(n)}$: sampled signal derivative
- $\Delta T = d\Delta t$: differentiation step (d is an integer greater or equal to 1, the case $d = 1$ means that the differentiation step is chosen equal to the sample step, i.e. the step between two consecutive samples).

This document only considers uniformly spaced data, i.e. Δt doesn't depend on *i*, and thus on time.

In some sections below (particularly in the *Signal and system approach* chapter), the following wordings and notations will equivalently be used:
- "input signal x_i" instead of "sampled signal f_i",
- "output signal y_i" instead of "$\tilde{f}_i$" or "$f_i^{(n)}$".

1.2 DEFINITIONS

Very generally, the moving-average filters and the finite difference schemes can be expressed through a linear constant-coefficient equation between their input and output, of the following type:

$$y_i = \sum_k a_k x_{i-k}$$

Kernel

The ***kernel*** of a scheme refers to the mass function $a(k) = a_k$, where k is thus an integer. In practice of course, only a few coefficients a_k are nonzero.
In other words, performing such a scheme is equivalent to applying the *kernel* function to each data point x_i of the time series. This means that all the samples of the time series are weighted using as weights the values of the *kernel* function.

Extent

In this document, the ***extent*** of a scheme will refer to the collection size of the samples which are encompassed in the scheme. The *extent* is simply the distance between the two lowest and highest non-zero a_k.
The *extent* can equally be expressed as:
- a sample number $k_{max} - k_{min} + 1$, or
- a time length $t_{k_{max}} - t_{k_{min}} = (k_{max} - k_{min})\Delta t$.

1.3 SUPPORTING EXAMPLE

In order to illustrate the principles developed in this document, the features under study will be illustrated by charts as far as possible.
For this purpose, the signal introduced in Figure 2 will be used as support throughout this document. Each matter under inspection will be exemplified in application to this signal.

Moreover, the first order backward difference with a 1 second differentiation step will be assumed to be the best available approximation of the exact derivative. This will be used as reference for comparison and will appear as light blue underlying curve in the charts when relevant (see Figure 3 for an example).

2 A MATTER OF CHOICES

2.1 TIME SHIFT VS. NOISE FILTERING

Particularly in the case of experimental data, substantial noise or imprecision may be present in the signal. And here is the crux of the matter: straightforward schemes with *short*-differentiation steps will amplify this noise, often so much that the result appears to be unusable (Chartrand). Sometimes, talking of noise amplifying is even an understatement: some schemes are affected in a way so sensitive by the high frequency noise in the input signal that they cause it literally to explode.

The Figure 2 illustrates real experimental data. The signal consists of deflection measurements – carried out with a position transducer – on a flexural loaded element. The data are acquired at a sampling period of 1 second. Apparently, the signal doesn't seem affected by any noise. Yet, this is merely an illusion…

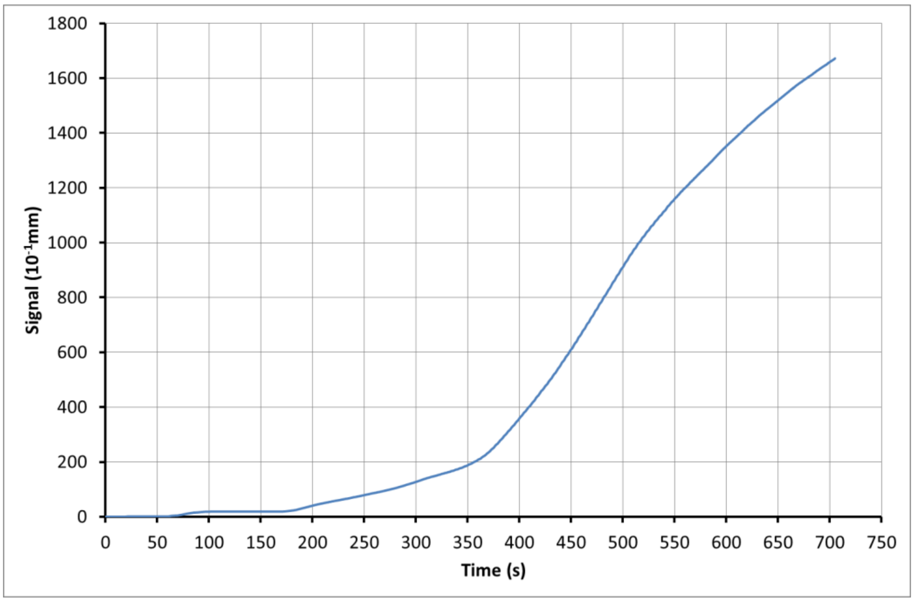


Figure 2 – Real experimental data of deflection measurements

As a first attempt, the differentiation has been processed through a first order backward difference with a 1 second differentiation step (i.e. the step between two consecutive samples). The resulting rate appears in blue curve on Figure 3 below. Its behaviour speaks for itself.

The next attempts make use of the same scheme but with a growing differentiation step (5, 20 and 60 seconds). While increasing the differentiation step has the positive effect of decreasing the noise, it also has two adverse effects: it produces a time shift compared with what would have been expected, and it introduces distortion by broadening and flattening the narrow features. These two negative effects can be summarized in two words: "poor fidelity".

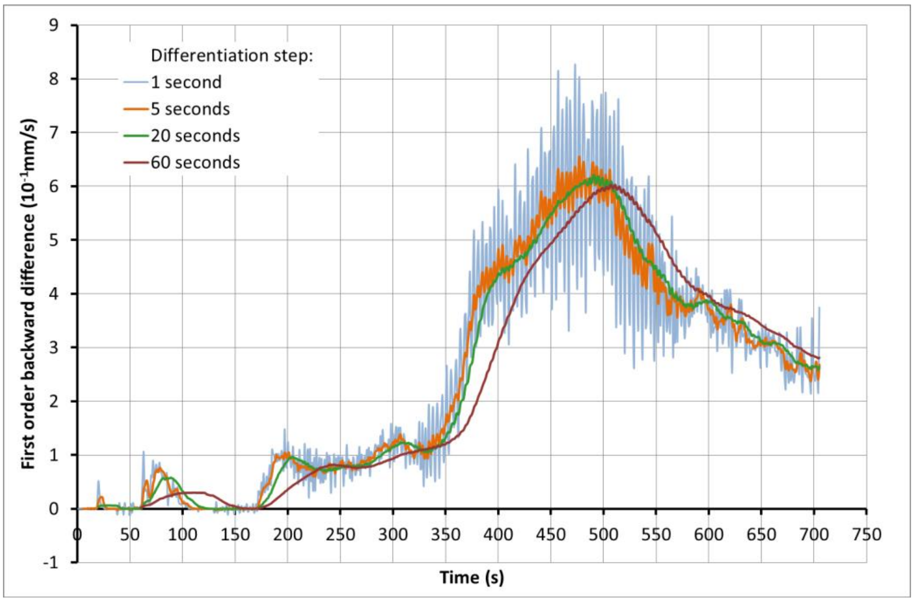


Figure 3 – Some backward differences attempts

The first lesson is that a compromise should probably be considered between noise reduction, and fidelity. The tools exposed below will help to understand the effect of the involved parameters on these two aspects. Solutions will be investigated to work around the problem by boosting both noise reduction and fidelity. This latter possibility will nevertheless be done at the expense of a truly real-time estimate.

2.2 CAUSALITY VS. NON-CAUSALITY

A system is causal if the output values of the system depend only on the present and the past input values, and do not depend on the future input values. A system is non-causal if its output values depend also on some future input values. Thus, a causal system is non-anticipative, this reflects the common fact that any effect must happen after the cause (Ghosh & Chakraborty, 2010).

In real-time processing applications, future input values are not yet known and cannot be predicted, real-time systems must thus be causal since they have no choice but to operate on current and past values of the signal (Semmlow, 2012).
However, in the case of off-line processing, i.e. if the data are already stored in a computer, then it is possible to use future signal values along with current and past values to compute an output signal (Semmlow, 2012). Such a situation is common in data processing.

As it will be exemplified in this document, all causal systems give rise to some time delay between their inputs and outputs. Eliminating the time shift inherent in causal systems is the primary motivation for using non-causal systems (Semmlow, 2012).

The second lesson is that a compromise should probably be considered between real-time processing, and eluding time-shift. In the present study, the interest will focus on real-time processing, and thus on causal schemes. Centered schemes – in spite of their non-causal nature – will also be encompassed for the comparison. Nevertheless, the exposed principles still apply for other non-causal schemes.

3 A SHORT PRESENTATION

3.1 MOVING AVERAGE FILTER

3.1.1 Definition

The moving average is the most common low-pass filter, mainly because it is the easiest filter to understand and use. In spite of its simplicity, the moving average filter is optimal for a common task: reducing random noise while retaining a sharp step response. This makes it the premier filter for time domain signals (Smith, 1997).
As the name implies, the moving average filter operates by averaging a number N of points from a raw signal f_i to produce each point in the filtered signal $\tilde{f}_i$. This set of N points forms the kernel of the filter, which is said to be an N-extent filter. In equation form, this is written:

$$\tilde{f}_i = \frac{1}{N}\sum_{j=0}^{N-1} f_{i-j+k}$$

where k is an integer parameter (Smith, 1997). The common practice limits the values of k to 3 possibilities:

- $k=0$, backward moving average: $\tilde{f}_i = \dfrac{f_{i-(N-1)} + f_{i-(N-2)} + \ldots + f_{i-1} + f_i}{N}$
- $k=N-1$, forward moving average: $\tilde{f}_i = \dfrac{f_i + f_{i+1} + \ldots + f_{i+(N-2)} + f_{i+(N-1)}}{N}$
- $k=\dfrac{N-1}{2}$ (if N is odd), centered moving average: $\tilde{f}_i = \dfrac{f_{i-\frac{N-1}{2}} + \ldots + f_i + \ldots + f_{i+\frac{N-1}{2}}}{N}$

The backward moving average makes use of present and past measurements, and is therefore a causal filter. The centered and forward moving averages make use of additional future measurements, and are therefore non-causal filters.

3.1.2 Observations

The backward moving average produces a relative time shift between the filtered signal and the raw signal, leading to a delayed estimation of the signal. On the contrary, the centered moving average doesn't produce any significant time shift. This backward delay effect occurs as soon as the signal is no longer constant, which is the case when dealing with real signals.
It will be shown that the resulting delay amounts to $\dfrac{(N-1)\Delta t}{2}$. In other words, the delay is half the filter extent. One can already easily observe that – if N is odd – $\dfrac{f_{i-(N-1)} + \ldots + f_i}{N}$ represents equivalently a backward moving average in t_i and a centered moving average in $t_{i-\frac{N-1}{2}}$, the first sampled time being delayed by $\dfrac{(N-1)\Delta t}{2}$ from the second one. As a consequence, the filtered signal cannot be known in real-

time, since the so-computed values are known with a lag of $\frac{\Delta T}{2}$ compared to the ideal filtered signal ("the calculated value at time *t* is the one that actually took place at $t-\frac{\Delta T}{2}$ ").

The figures below illustrate the effect of using both methods on a discrete-time signal with a sample step of *0,2 time unit*. This signal has been noised by evenly distributed random values.

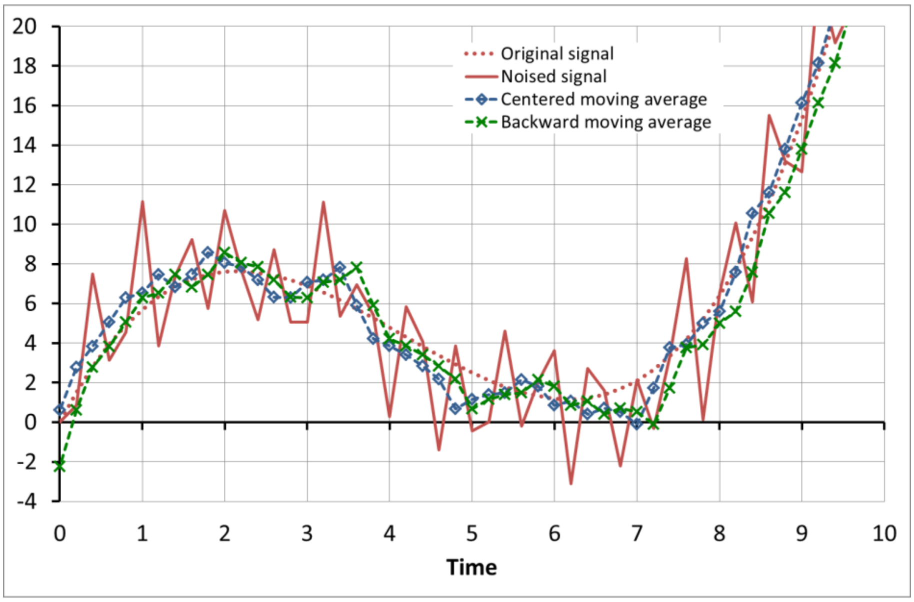


Figure 4 – Moving average filter with a *3*-point extent
(the foreseen delay of *0,2 time unit* is perceived)

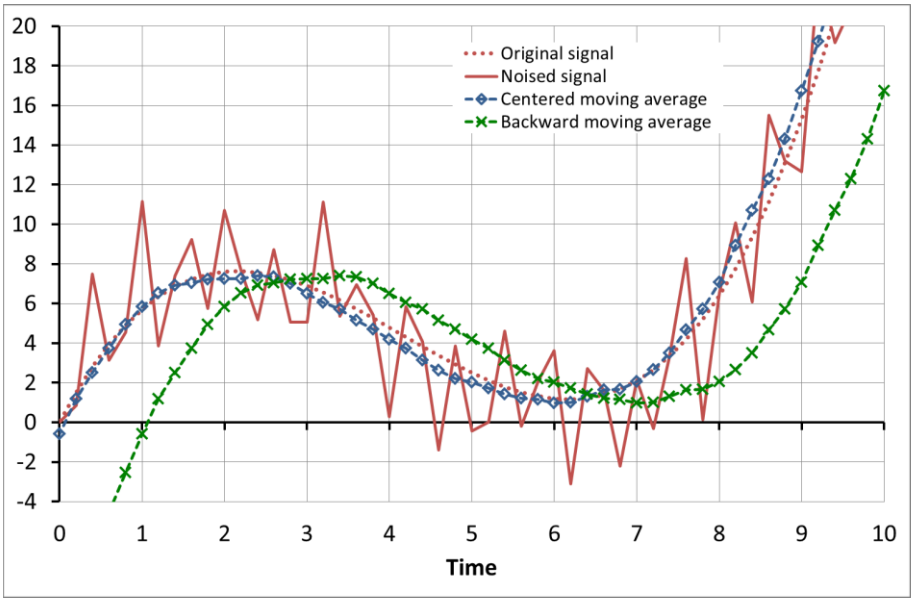


Figure 5 – Moving average filter with an *11*-point extent
(the foreseen delay of *1 time unit* appears clearly)

3.2 FINITE DIFFERENCES

3.2.1 Definition

Any continuous differentiable function f can be expressed into a Taylor series at any point a:

$$f(t)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(t-a)^n=f(a)+\frac{f^{(1)}(a)}{1!}(t-a)+\frac{f^{(2)}(a)}{2!}(t-a)^2+\frac{f^{(3)}(a)}{3!}(t-a)^3+\dots$$

The application of such Taylor expansion at point t_i, using the here considered sampled values, gives:

$$f_{i+kd}=f_i+f_i^{(1)}k\Delta T+\frac{f_i^{(2)}}{2}(k\Delta T)^2+\frac{f_i^{(3)}}{6}(k\Delta T)^3+\frac{f_i^{(4)}}{24}(k\Delta T)^4+\mathcal{O}(\Delta T^5)$$

For example:

- $f_{i+d}=f_i+f_i^{(1)}\Delta T+f_i^{(2)}\frac{\Delta T^2}{2}+f_i^{(3)}\frac{\Delta T^3}{6}+f_i^{(4)}\frac{\Delta T^4}{24}+\mathcal{O}(\Delta T^5)$
- $f_{i-d}=f_i-f_i^{(1)}\Delta T+f_i^{(2)}\frac{\Delta T^2}{2}-f_i^{(3)}\frac{\Delta T^3}{6}+f_i^{(4)}\frac{\Delta T^4}{24}+\mathcal{O}(\Delta T^5)$
- $f_{i-2d}=f_i-f_i^{(1)}2\Delta T+f_i^{(2)}2\Delta T^2-f_i^{(3)}\frac{4\Delta T^3}{3}+f_i^{(4)}\frac{2\Delta T^4}{3}+\mathcal{O}(\Delta T^5)$
- $f_{i-3d}=f_i-f_i^{(1)}3\Delta T+f_i^{(2)}\frac{9\Delta T^2}{2}-f_i^{(3)}\frac{9\Delta T^3}{2}+f_i^{(4)}\frac{27\Delta T^4}{8}+\mathcal{O}(\Delta T^5)$
- …

Appropriate linear combinations of these Taylor expansions give the finite difference formulas and their accuracies. Here are some examples, limited to the ones that will be used further.

First derivatives

- backward difference (causal):
  - $f_i^{(1)}=\frac{f_i-f_{i-d}}{\Delta T}\underbrace{\left\{+f_i^{(2)}\frac{\Delta T}{2}-f_i^{(3)}\frac{\Delta T^2}{6}+f_i^{(4)}\frac{\Delta T^3}{24}+\mathcal{O}(\Delta T^4)\right\}}_{truncation\,error\,\varepsilon=\mathcal{O}(\Delta T)}$ (first order scheme)
  - $f_i^{(1)}=\frac{3f_i-4f_{i-d}+f_{i-2d}}{2\Delta T}\underbrace{\left\{+f_i^{(3)}\frac{\Delta T^2}{3}-f_i^{(4)}\frac{\Delta T^3}{4}+\mathcal{O}(\Delta T^4)\right\}}_{truncation\,error\,\varepsilon=\mathcal{O}(\Delta T^2)}$ (second order scheme)
  - $f_i^{(1)}=\frac{11f_i-18f_{i-d}+9f_{i-2d}-2f_{i-3d}}{6\Delta T}\underbrace{\left\{+f_i^{(4)}\frac{\Delta T^3}{4}+\mathcal{O}(\Delta T^4)\right\}}_{truncation\,error\,\varepsilon=\mathcal{O}(\Delta T^3)}$ (third order scheme)
- centered difference (non-causal):
  - $f_i^{(1)}=\frac{f_{i+d}-f_{i-d}}{2\Delta T}\underbrace{\left\{-f_i^{(3)}\frac{\Delta T^2}{6}+\mathcal{O}(\Delta T^4)\right\}}_{truncation\,error\,\varepsilon=\mathcal{O}(\Delta T^2)}$ (second order scheme)

Second derivatives

- backward difference (causal):
  - $$f_i^{(2)} = \frac{f_i - 2f_{i-d} + f_{i-2d}}{\Delta T^2} \underbrace{\left\{ + f_i^{(3)} \Delta T - f_i^{(4)} \frac{7\Delta T^2}{12} + \mathcal{O}(\Delta T^3) \right\}}_{truncation error\ \varepsilon = \mathcal{O}(\Delta T)} \text{ (first order scheme)}$$

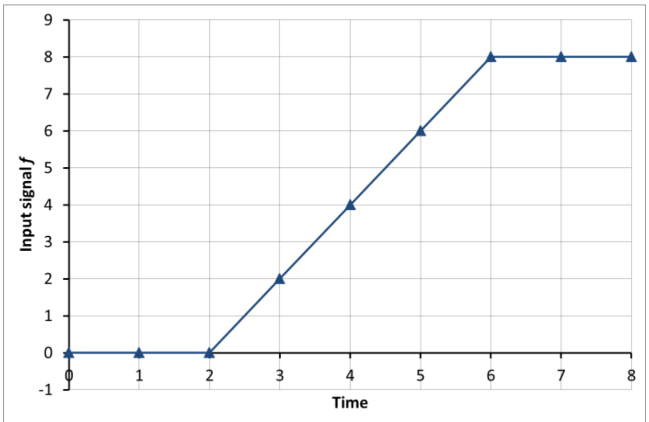
Third derivatives

- backward difference (causal):
  - $$f_i^{(3)} = \frac{f_i - 3f_{i-d} + 3f_{i-2d} - f_{i-3d}}{\Delta T^3} \underbrace{\left\{ + f_i^{(4)} \frac{3\Delta T}{2} + \mathcal{O}(\Delta T^2) \right\}}_{truncation error\ \varepsilon = \mathcal{O}(\Delta T)} \text{ (first order scheme)}$$

Note that there is no formal limitation to the possibilities. As an example, asymmetric combination of backward and forward points leads to the uncentered scheme (non-causal):

$$f_i^{(1)} = \frac{2f_{i+d} + 3f_i - 6f_{i-d} + f_{i-2d}}{6\Delta T} \underbrace{\left\{ - f_i^{(4)} \frac{\Delta T^3}{12} + \mathcal{O}(\Delta T^4) \right\}}_{truncation error\ \varepsilon = \mathcal{O}(\Delta T^3)} \text{ (third order scheme)}$$

The backward difference schemes make use of present and past measurements, and are therefore causal differentiation methods. The centered and forward difference schemes make use of additional future measurements, and are therefore non-causal differentiation methods. The figure below illustrates the effect of using both methods on a piecewise linear signal ($f = 2t - 4$ for $2 \le t \le 6$).



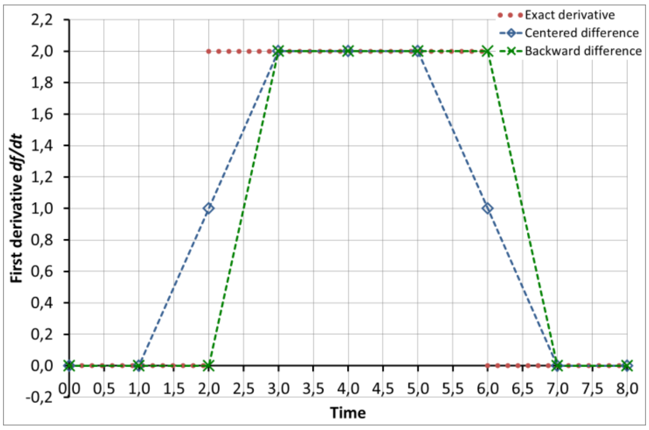


Figure 6 – Comparison of backward and centered difference schemes on a linear signal

3.2.2 Observations

The backward differences produce a relative time shift between the calculated derivative and the exact derivative, leading to a delayed estimation of the derivative. On the contrary, the centered difference doesn't produce any significant time shift. This backward delay effect occurs as soon as the polynomial order of the signal is greater than the difference scheme order (a quadratic signal for a first order scheme is an example), which is the case when dealing with real signals.

It will be shown that the resulting delay for the first order backward scheme amounts to $\frac{\Delta T}{2}$. In other words, the delay is half the differentiation step. One can already easily observe that $\frac{f_i - f_{i-2k}}{2k\Delta t}$ represents equivalently a backward difference in t_i and a centered difference in t_{i-k}, the first sampled time being delayed by $k\Delta t = \frac{\Delta T}{2}$ from the second one. As a consequence, the calculated derivative cannot be known in real-time, since the so-computed values are known with a lag of $\frac{\Delta T}{2}$ compared to the exact derivative ("the calculated value at time t is the one that actually took place at $t - \frac{\Delta T}{2}$").

The figures below illustrate the effect of using both methods on two examples (a quadratic signal $f = t^2$, a cubic signal $f = 0{,}2t^3 - 2{,}5t^2 + 8t$, and a real-life signal) with a sample step of *1 time unit*.

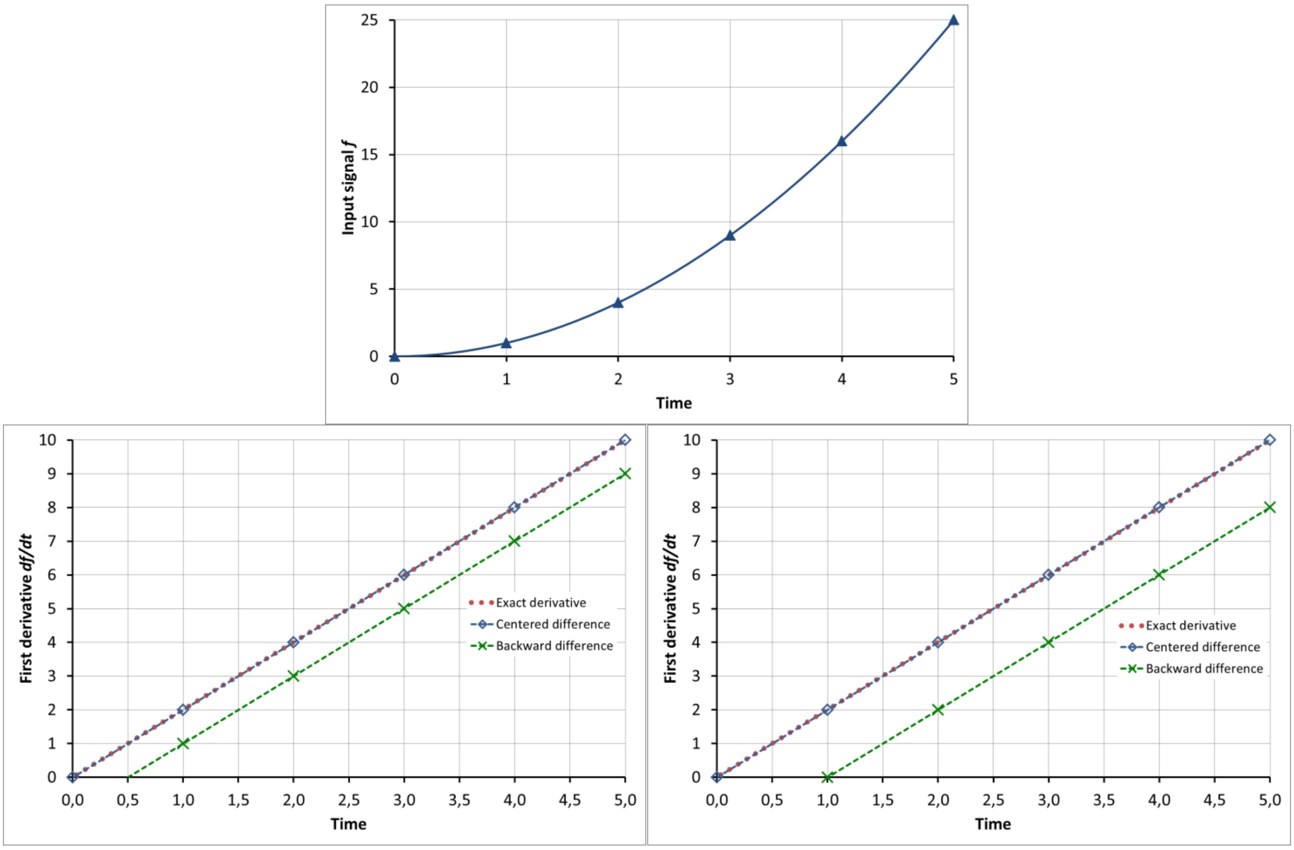


Figure 7 – Comparison of backward and centered difference schemes on a quadratic signal
Left: differentiation step of *1 time unit* ($d = 1$) Right: differentiation step of *2 time units* ($d = 2$)
(the foreseen delay of *0,5 time unit* appears clearly) (the foreseen delay of *1 time unit* appears clearly)

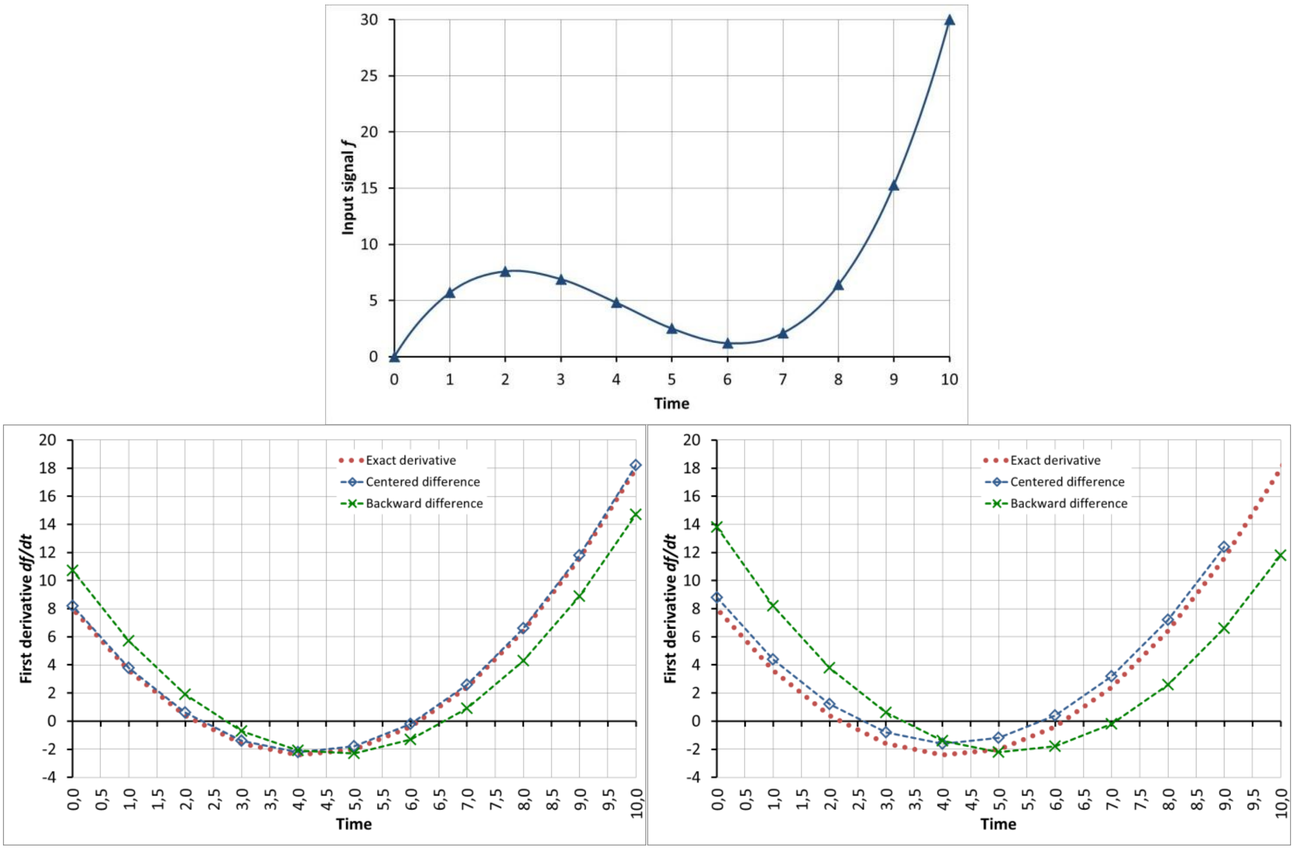


Figure 8 – Comparison of backward and centered difference schemes on a cubic signal
Left: differentiation step of *1 time unit* ($d = 1$) Right: differentiation step of *2 time units* ($d = 2$)
(the foreseen delay of *0,5 time unit* appears clearly) (the foreseen delay of *1 time unit* appears clearly)

Finally, the chart below compares backward difference schemes of 1st, 2nd and 3rd orders. The time-shift effect seems to reduce when the scheme order increases, at the expense of a less efficient noise filtering and the occurrence of a specific kind of distortion. This behaviour will be demonstrated below.

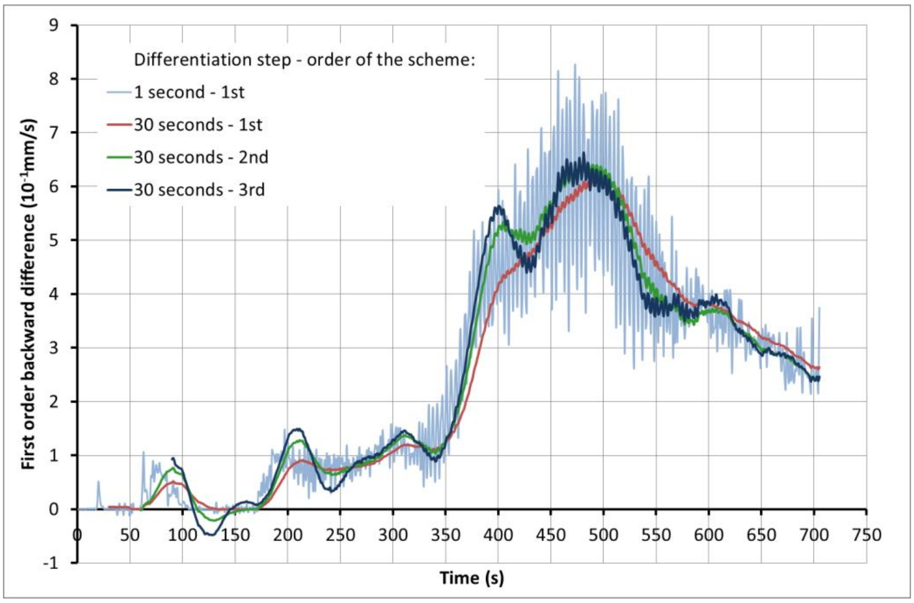


Figure 9 – Comparison of backward difference schemes of different orders

3.2.3 A notable property

Applying the first order backward difference of a given discrete-time signal by using consecutive data points (differentiation step chosen equal to the sample step, $d = 1$) gives the rate of the signal at each sampled point *i* by:

$$f_i^{(1)} = \frac{f_i - f_{i-1}}{\Delta t}$$

These values could then be passed through a backward moving average filter. An *N*-point extent filter gives the filtered rate of the signal at each sampled point *i* by:

$$\tilde{f}_i^{(1)} = \frac{1}{N}\sum_{j=0}^{N-1} f_{i-j}^{(1)} = \frac{1}{N}\sum_{j=0}^{N-1}\frac{f_{i-j} - f_{i-j-1}}{\Delta t} = \frac{f_i - f_{i-N}}{N\Delta t}$$

In other words, applying an *N*-differentiation step first order backward difference $(\Delta T = N\Delta t)$ is equivalent to applying successively an *N*-extent backward moving average filter on a *one*-differentiation step first order backward difference $(\Delta T = \Delta t)$.

As a matter of fact, this example can be generalized: all schemes with *multiple*-differentiation steps implicitly incorporate a low-pass filtering effect. This will be exemplified below.

4 GRAPHICAL APPROACH

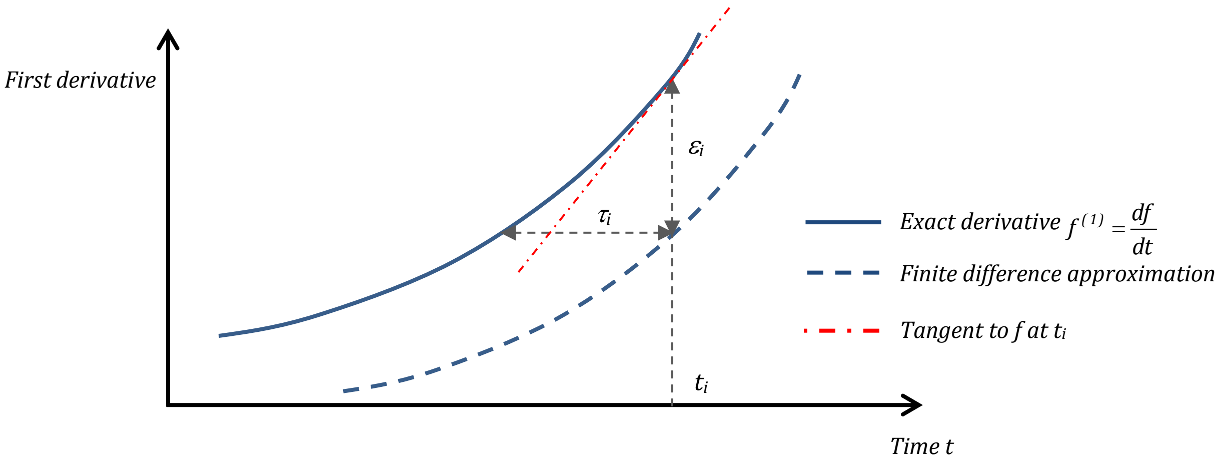


Figure 10 – Graphical approach

Denoting by ε_i the truncation error of the considered finite difference scheme at point t_i, and by τ_i the associated numerical time delay, this latter delay can be evaluated by the following approximation:

$$\tau_i \approx \frac{\varepsilon_i}{f_i^{(2)}}$$

A negative delay ($\tau_i \prec 0$) means that the finite difference approximation is time-advanced compared to the exact derivative, while a positive delay ($\tau_i \succ 0$) means that the finite difference approximation is time-delayed.

The application of this approximation to the first derivative schemes exposed above gives the delays below:

- backward difference:
  - □ $\tau_i \approx \frac{\Delta T}{2} - \frac{f_i^{(3)}}{f_i^{(2)}}\frac{\Delta T^2}{6} + \frac{f_i^{(4)}}{f_i^{(2)}}\frac{\Delta T^3}{24} + \mathcal{O}(\Delta T^4)$ (first order scheme)
  - □ $\tau_i \approx \frac{f_i^{(3)}}{f_i^{(2)}}\frac{\Delta T^2}{3} - \frac{f_i^{(4)}}{f_i^{(2)}}\frac{\Delta T^3}{4} + \mathcal{O}(\Delta T^4)$ (second order scheme)
  - □ $\tau_i \approx \frac{f_i^{(4)}}{f_i^{(2)}}\frac{\Delta T^3}{4} + \mathcal{O}(\Delta T^4)$ (third order scheme)
- centered difference:
  - □ $\tau_i \approx -\frac{f_i^{(3)}}{f_i^{(2)}}\frac{\Delta T^2}{6} + \mathcal{O}(\Delta T^4)$ (second order scheme)

Actually, one should realize that these evaluations of τ_i are only estimations based on a linearized approximation using the tangent to f at t_i. As a consequence, the resulting error of this approximation affects the terms in ΔT^n of orders $n \succ 1$. In other words, only the first term of the first order scheme above makes sense; all other terms are affected by that error, in such a way that they no longer mean anything and thus prove impossible to interpret.

For this reason, the analysis will now focus on the only relevant relation $\tau_i \approx \frac{\Delta T}{2}$.

The first order backward difference scheme appears to be mainly biased by a constant shift, amounting to $\frac{\Delta T}{2}$. This first order time-error is ever positive, meaning that this scheme always time-delays the calculated derivative. Systematically shift the calculated derivative by this constant value is the smartest correction that can be made, and it should be done! In practice, this is done by subtracting $\frac{\Delta T}{2}$ from the discrete time values. The efficiency on this first order correction is illustrated below on two examples.

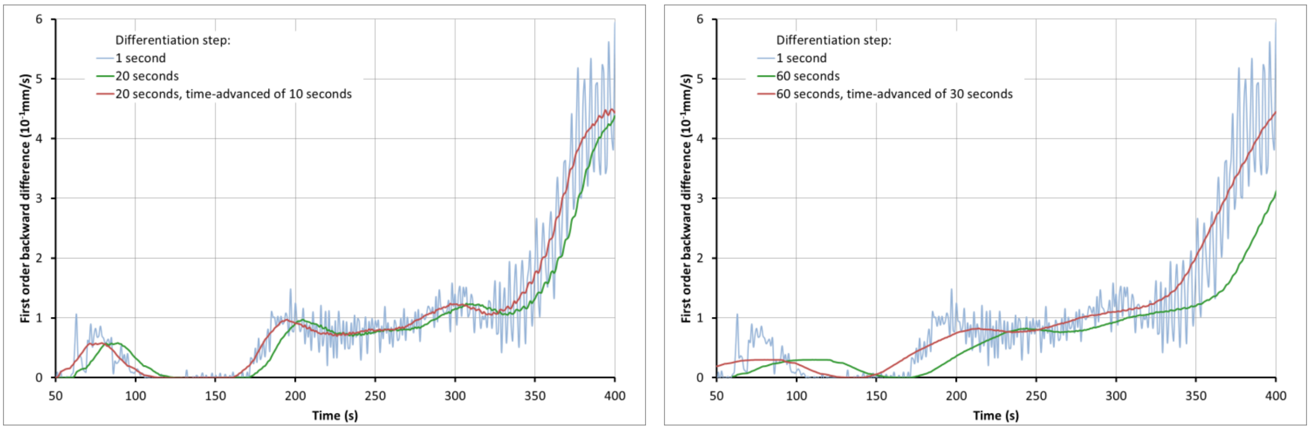


Figure 11 – First order backward scheme with first order correction

The same reasoning can be carried out for higher derivatives. For example, the time delay for the first order second derivative is evaluated by:

$$\tau_i \approx \frac{\varepsilon_i}{f_i^{(3)}}$$

and leads to:

$$\tau_i \approx \Delta T$$

The analysis then follows as above. The first order backward difference scheme appears to be mainly biased by a constant shift, amounting to ΔT. This first order time-error is ever positive, meaning that this scheme always time-delays the calculated derivative.

5 SIGNAL AND SYSTEM APPROACH

5.1 TIME DOMAIN

A signal is a description of how one parameter varies with another parameter. A system is any process that produces an output signal in response to an input signal (Smith, 1997). As the present study deals with sampled signals, only will be considered here discrete systems and discrete-time signals.

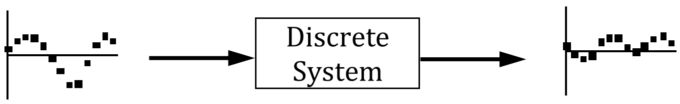
Example:

The measured deflection $f_i = f(t_i)$ is a discrete-time signal.

The first derivative backward difference $f_i^{(1)} = \frac{f_i - f_{i-d}}{\Delta T}$ is a system that produces a rate of deflection (output signal $f_i^{(1)}$) from a deflection (input signal f_i).

Any system can be described by an operator $\mathcal{H}$ which transforms an input sequence into an output sequence:

$$y_i = \mathcal{H}\{x_i\}$$



x_i $\quad$ $y_i = \mathcal{H}\{x_i\}$ $\quad$ y_i

Figure 12 – Representation of a system (Smith, 1997)

Defining the unit impulse as the delta function

$$\delta_i = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases}$$

then any signal can be expressed as a linear combination of suitably weighed and shifted impulses. In this case, the weights are nothing but the signal values themselves (Prandoni & Vetterli, 2008). This is called the impulse decomposition:

$$x_i = \sum_{k=-\infty}^{\infty} x_k \delta_{i-k}$$

The impulse response h_i of a system is the signal that exits the system when a delta function (unit impulse) is the input (Smith, 1997):

$$h_i = \mathcal{H}\{\delta_i\}$$

LTI systems

A system is linear if

$$\mathcal{H}\{ax_{1,i} + bx_{2,i}\} = a\mathcal{H}\{x_{1,i}\} + b\mathcal{H}\{x_{2,i}\}$$

for any two sequences $x_{1,i}$ and $x_{2,i}$ any two scalars a, b.

A system is time-invariant if

$$y_i = \mathcal{H}\{x_i\} \Leftrightarrow y_{i-k} = \mathcal{H}\{x_{i-k}\}$$

This time-invariance is important because it means that the characteristics of the system do not change with time (or whatever the input signal happens to be). In other words, if a "blip" in the input causes a "blop" in the output, then another "blip" at any other time will cause an identical "blop" (Smith, 1997).

When linearity and time-invariance a met simultaneously, the system is said to be a linear time-invariant system, more commonly referred as LTI. These two properties taken together have an incredibly powerful consequence on a system's behaviour. Indeed, an LTI system turns out to be completely characterized by its impulse response h_i : this is all one needs to know to determine the system's output for any input signal (Prandoni & Vetterli, 2008). In other words, knowing its impulse response h_i means knowing everything about the system. This immediately follows from the here above definitions and properties that allow expressing the output of an LTI system as:

$$y_i = \sum_{k=-\infty}^{\infty} x_k h_{i-k}$$

where the impulse response h_i is invariant and thus unique. Such summation is nothing but the well-known convolution sum of x_i and h_i and is denoted by the operator *, so that the above relation can be shorthanded to:

$$y_i = x_i * h_i$$

The way of understanding how an LTI system changes an input signal into an output signal can be stated as follows. First, the input signal can be decomposed into a set of impulses, each of which can be viewed as a scaled and shifted delta function. Second, the output resulting from each impulse is a scaled and shifted version of the impulse response. Third, the overall output signal can be found by adding these scaled and shifted impulse responses (Smith, 1997).

5.2 FREQUENCY DOMAIN

Fourier analysis is a family of mathematical techniques, all based on decomposing signals into sinusoids and cosinusoids and thus revealing the frequency spectrum of the signals. The signals that can be encountered in the present study are aperiodic discrete type (signals only defined at discrete points between positive and negative infinity, and do not repeat themselves in a periodic sequence). Amongst the different types of Fourier transform, the Discrete-Time Fourier Transform (DTFT) is the one used for such signal type (Smith, 1997).

5.2.1 Definition

The complex DTFT transform $X_\omega = \mathcal{F}\{x_i\}$ changes an input signal x_i into an output signal X_ω (the frequency spectrum of the input signal) through the decomposition equation (Mandal & Asif, 2007):

$$X_\omega = \sum_{i=-\infty}^{\infty} x_i e^{-j\omega i}$$

where $e^{jx} = cos(x) + j\,sin(x)$ (Euler's relation) and $j = \sqrt{-1}$. The DTFT decomposes the time domain signal into the frequency domain signal, this one containing the amplitudes and the frequencies of the frequency spectrum (component sine and cosine waves) (Smith, 1997).
Inversely, the inverse DTFT transform $x_i = \mathcal{F}^{-1}\{X_\omega\}$ changes an input signal X_ω into an output signal x_i through the synthesis equation (Mandal & Asif, 2007):

$$x_i = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_\omega e^{j\omega i} d\omega$$

The inverse DTFT synthesizes the time domain signal from the frequency domain signal.

In its rectangular form, namely $X_\omega = \Re(X_\omega) + j\Im(X_\omega)$, the complex DTFT can also be written as the combination of a real and an imaginary part.
In its polar form, namely $X_\omega = |X_\omega| e^{j\theta_\omega}$, the complex DTFT can also be written as the combination of a magnitude and a phase:

$$magnitude = |X_\omega| = \sqrt{\Re^2(X_\omega) + \Im^2(X_\omega)}$$

$$phase = \theta_\omega = arctan\left(\frac{\Im(X_\omega)}{\Re(X_\omega)}\right)$$

5.2.2 Some useful properties

- It directly follows from its definition that the DTFT is a linear operator.

- It directly follows from its definition that the DTFT of a real-valued signal is conjugate-symmetric $(X_\omega = X^*_{-\omega})$, resulting in:

$$\Re(X_\omega) = \Re(X_{-\omega}) \qquad \Im(X_\omega) = -\Im(X_{-\omega})$$
$$|X_\omega| = |X_{-\omega}| \qquad \theta_\omega = -\theta_{-\omega}$$

- A fundamental property used in signal processing is that Fourier transform of a convolution sum is equal to the product of the Fourier transform. In other words, the convolution sum in the time domain corresponds to multiplication in the frequency domain. It has been shown above that LTI systems are fully described in the time domain by the convolution sum $y_i = x_i * h_i$. As a consequence, an LTI system can also be fully described in the frequency domain by the product $Y_\omega = X_\omega . H_\omega$, where $H_\omega = \mathcal{F}\{h_i\}$ is called the "frequency response" (i.e. the frequency response is the Fourier transform of the impulse response). Both the impulse response and the frequency response contain complete information about the system.

- A time domain signal and its associated frequency domain signal form a pair. The literature lists tables of pairs for most encountered common signals (as examples, see (Prandoni & Vetterli, 2008), (Mandal & Asif, 2007), or (Rao Yarlagadda, 2010)). The further use of the impulse decomposition together with the two above properties lead the interest on the particular following pair:

$$\mathcal{F}\{\delta_{i-k}\} = e^{-j\omega k}$$

- The Fourier equations are conceptual representations of discrete time signals that rely on the notion of dimensionless frequencies ω. The absence of a physical dimension for time has the happy consequence that all discrete time signal Fourier processing become indifferent to the underlying physical nature of the actual signals. This dimensionless abstraction, however, is a drawback from the point of view of intuition because of usual familiarity with signals in the real world for which time is expressed in seconds and frequency is expressed in Hertz (Prandoni & Vetterli, 2008). The precise, formal link between "real world" dimensional frequency f_{Hz} (in Hertz) and "Fourier" dimensionless frequency ω is given by:

$$f_{Hz} = \frac{\omega}{2\pi\Delta t}$$

and its related angular frequency ω_{Hz} by:

$$\omega_{Hz} = \frac{\omega}{\Delta t}$$

Remembering that $t_i = i\Delta t$, the arguments ωi of Fourier equations become

$$\omega i = \omega t_i / \Delta t = \omega_{Hz} t_i$$

i.e. the well-known physical form.

5.2.3 Exact derivatives in frequency domain

In the framework of the study of the properties of calculated derivatives by the finite difference method, it would be very useful to compare them with the exact derivatives $\frac{d^n x(t)}{dt^n}$, where $x(t)$ refers to the continuous-time signal from which the discrete-time signal x_i (under consideration in the present study) has been sampled.

The exact derivative being only defined in the continuous-time space – and not in the discrete-time space – the Continuous-Time Fourier Transform (CTFT) is the relevant Fourier transform to use. Without going into detail, the CTFT can be seen as the limit of the DTFT (Discrete-Time Fourier Transform) when $\Delta t \to 0$. As the DTFT, the CTFT reveals the frequency spectrum. This full equivalence allows drawing a direct comparison between calculated and exact derivatives in the frequency domain.
The CTFT provides the useful pair

$$\mathcal{F}\left\{\frac{d^n x(t)}{dt^n}\right\} = (j\omega)^n X_\omega$$

The frequency response of an ideal differentiator is thus

$$H_\omega^{Ideal\,dif.} = (j\omega)^n$$

<u>Examples:</u>

The frequency response of the first derivative system is

$$H_\omega^{Ideal\,dif.} = j\omega = |\omega| e^{j\frac{\pi}{2}}$$

The frequency response of the second derivative system is

$$H_\omega^{Ideal\,dif.} = -\omega^2 = \omega^2 e^{j\pi}$$

The ideal differentiator has a frequency response that increases with frequency; therefore it greatly amplifies high-frequency noises. In practice, when dealing with noisy signals, one would be readily choose a low-pass differentiator rather than a full-pass one (Luo, Ying, & Bai, 2005).

5.3 LINEAR CONSTANT-COEFFICIENT SCHEMES

The moving-average filters and the finite differences share the interesting property of being LTI systems since their input and output are related through a linear constant-coefficient equation of the following type:

$$y_i = \sum_{k=k_1}^{k_2} a_k x_{i-k}$$

whose impulse response is thus:

$$h_i = \sum_{k=k_1}^{k_2} a_k \delta_{i-k}$$

In practice of course, only a few coefficients a_k are nonzero.

Using the properties of the Fourier operator, the frequency response $H_\omega = \mathcal{F}\{h_i\}$ of such linear constant-coefficient equation can be expressed as:

$$H_\omega = \sum_{k=k_1}^{k_2} a_k e^{-j\omega k}$$

APPLICATIONS

Backward moving average on N points

$$y_i = \frac{1}{N}\sum_{k=0}^{N-1} x_{i-k} = \frac{x_{i-(N-1)} + x_{i-(N-2)} + \ldots + x_{i-1} + x_i}{N}$$

In this case:

$$a_k = \begin{cases} 1/N & 0 \le k \le N-1 \\ 0 & else \end{cases}$$

The frequency response is:

$$H_\omega = \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\omega k}$$

The above summation represents a geometric progression series, whose general formula is:

$$\sum_{n=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

The frequency response becomes:

$$H_\omega = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

Using some manipulations and the Euler's relation for sine $sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$ leads to:

$$H_\omega = \left| \frac{1}{N} \frac{sin\left(\frac{\omega N}{2}\right)}{sin\left(\frac{\omega}{2}\right)} \right| e^{-j\frac{\omega(N-1)}{2}}$$

First derivative backward difference (first order scheme)

$$y_i = \frac{x_i - x_{i-d}}{\Delta T}$$

In this case:

$$a_k = \begin{cases} 1/\Delta T & k = 0 \\ -1/\Delta T & k = d \\ 0 & else \end{cases}$$

The frequency response is:

$$H_\omega = \frac{1-e^{-j\omega d}}{\Delta T}$$

Using some manipulations and the Euler's relation for sine $sin(x)=\frac{e^{jx}-e^{-jx}}{2j}$ leads to:

$$H_\omega=\left|\frac{2}{\Delta T}sin\left(\frac{\omega d}{2}\right)\right|e^{-j\frac{(\omega d-\pi)}{2}}$$

First derivative backward difference (second order scheme)

$$y_i=\frac{3x_i-4x_{i-d}+x_{i-2d}}{2\Delta T}$$

In this case:

$$a_k=\begin{cases}3/2\Delta T & k=0\\ -4/2\Delta T & k=d\\ 1/2\Delta T & k=2d\\ 0 & else\end{cases}$$

The frequency response is:

$$H_\omega=\frac{3-4e^{-j\omega d}+e^{-j2\omega d}}{2\Delta T}$$

$$H_\omega=\frac{(1-cos(\omega d))^2}{\Delta T}+j\frac{sin(\omega d)(2-cos(\omega d))}{\Delta T}$$

Using some manipulations and fundamental trigonometric relations leads to:

$$H_\omega=\frac{\sqrt{3cos^2(\omega d)-8cos(\omega d)+5}}{\Delta T}e^{j\,arctan\left(\frac{sin(\omega d)(2-cos(\omega d))}{(1-cos(\omega d))^2}\right)}$$

First derivative backward difference (third order scheme)

$$y_i=\frac{11x_i-18x_{i-d}+9x_{i-2d}-2x_{i-3d}}{6\Delta T}$$

In this case:

$$a_k=\begin{cases}11/6\Delta T & k=0\\ -18/6\Delta T & k=d\\ 9/6\Delta T & k=2d\\ -2/6\Delta T & k=3d\\ 0 & else\end{cases}$$

The frequency response is:

$$H_\omega=\frac{11-18e^{-j\omega d}+9e^{-j2\omega d}-2e^{-j3\omega d}}{6\Delta T}$$

or

$$H_\omega=\frac{1-6cos(\omega d)+9cos^2(\omega d)-4cos^3(\omega d)}{3\Delta T}+j\frac{sin(\omega d)(8-9cos(\omega d)+4cos^2(\omega d))}{3\Delta T}$$

No interesting simplification could be found for this expression, so it will be used as it is.

First derivative centered difference (second order scheme)

$$y_i = \frac{x_{i+d} - x_{i-d}}{2\Delta T}$$

In this case:

$$a_k = \begin{cases} 1/2\Delta T & k=-d \\ -1/2\Delta T & k=d \\ 0 & else \end{cases}$$

The frequency response is:

$$H_\omega = \frac{e^{j\omega d} - e^{-j\omega d}}{2\Delta T}$$

Using some manipulations and the Euler's relation for sine $sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$ leads to:

$$H_\omega = \left| \frac{1}{\Delta T} sin(\omega d) \right| e^{j\frac{\pi}{2}}$$

Second derivative backward difference (first order scheme)

$$y_i = \frac{x_i - 2x_{i-d} + x_{i-2d}}{\Delta T^2}$$

In this case:

$$a_k = \begin{cases} 1/\Delta T^2 & k=0 \\ -2/\Delta T^2 & k=d \\ 1/\Delta T^2 & k=2d \\ 0 & else \end{cases}$$

The frequency response is:

$$H_\omega = \frac{1 - 2e^{-j\omega d} + e^{-j2\omega d}}{\Delta T^2}$$

Using then some manipulations and fundamental trigonometric relations leads to:

$$H_\omega = \frac{4}{\Delta T^2} sin^2\left(\frac{\omega d}{2}\right) e^{-j(\omega d - \pi)}$$

5.4 SPECTRUM ANALYSIS

Let's resume two main concepts.

- An LTI system is fully described in the frequency domain by the product $Y_\omega = X_\omega . H_\omega$. The quantities X_ω and Y_ω are the DTFT of the input and output signals, while the frequency response H_ω is the DTFT of the impulse response of the operator describing the LTI system that produces the output signal in response to the input signal.
- A Fourier transform generates a spectrum in the frequency domain – $W_\omega = \mathcal{F}\{w_i\}$ – from a signal or an operator w_i. This spectrum – function of ω – is a complete, alternative representation of the signal or operator, and the analysis of the spectrum reveals the fundamental information required to characterize and classify the signal or operator in the frequency domain (Prandoni & Vetterli, 2008).

The relation $Y_\omega = X_\omega . H_\omega$ highlights the way the frequency response H_ω acts on the input signal: through this simple multiplication, H_ω modulates X_ω into Y_ω. The spectrum analysis of H_ω will thus inform how the LTI system modify the input signal into the output signal at each angular frequency ω.

Since the transform values are complex numbers, it is customary to separately analyse their magnitude and their phase. As a reminder, the complex frequency response can be written in its polar form $H_\omega = |H_\omega| e^{j\theta_\omega}$ where:

$$magnitude = |H_\omega| = \sqrt{\Re^2(H_\omega) + \Im^2(H_\omega)}$$

$$phase = \theta_\omega = arctan\left(\frac{\Im(H_\omega)}{\Re(H_\omega)}\right)$$

The relation between dimensionless and dimensional frequencies $f_{Hz} = \frac{\omega}{2\pi\Delta t}$ and angular frequencies $\omega i = \omega t_i / \Delta t = \omega_{Hz} t_i$ will also be used in the analysis.

Note that in the charts here below, the axes have been made intentionally dimensionless. This choice is more convenient for the analysis.

5.4.1 Magnitude spectrum

From the perspective of the Fourier representation as being a sum of sine and cosine waves, the magnitude spectrum defines the inherent power produced by each of the waves (Prandoni & Vetterli, 2008). The magnitude is thus related to the energy distribution of the operator in the frequency domain. By its "scaling" action, the magnitude $|H_\omega|$ of an operator will expand or attenuate the input signal, as a function of ω. This corresponds to the filtering effect of the operator.

According to the way the magnitude spectrum affects the signal, the operator filtering effect can be classified into broad categories (lowpass, highpass, bandpass, … operators).

The frequency interval (or intervals) for which the magnitude of the frequency response is zero (or practically negligible) is called the stopband, or the cuttoff frequency. Conversely, the frequency interval (or intervals) for which the magnitude is non-negligible is called the passband (Prandoni & Vetterli, 2008).

In the spectra below, the curve of the frequency response of ideal differentiators ($H_\omega^{Idealdif.} = (j\omega)^n$) will be shown for comparison.

APPLICATIONS

Backward moving average on *N* points

$$|H_\omega| = \left| \frac{1}{N} \frac{sin\left(\frac{\omega N}{2}\right)}{sin\left(\frac{\omega}{2}\right)} \right|$$

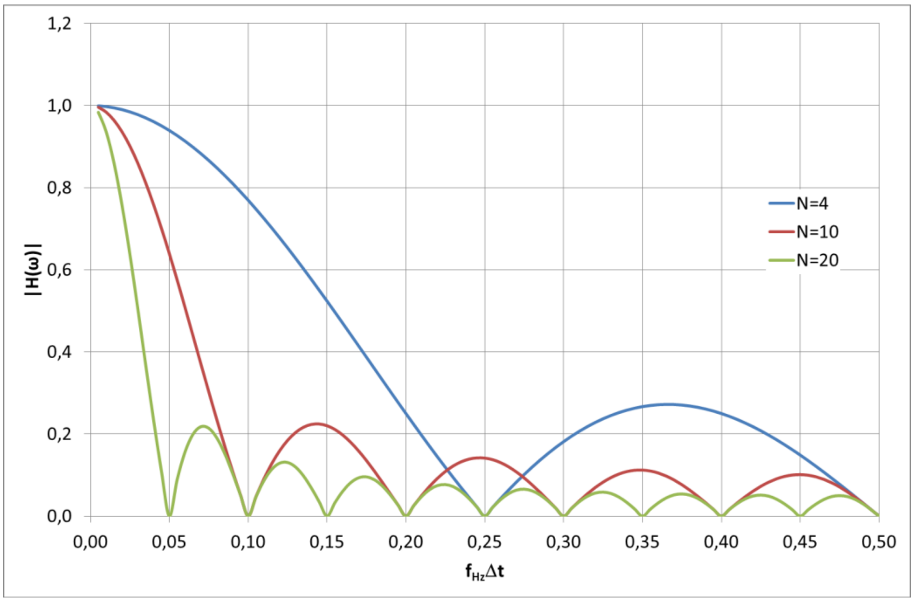


Figure 13 – Backward moving average with an N-extent

The efficiency of the high frequency attenuation is rather poor, and the stopband is not so clear. The moving average filter cannot effectively separate one band of frequencies from another. But good performance in the time domain results in poor performance in the frequency domain, and vice versa. In short, the moving average should be considered as a very good smoothing filter (the action in the time domain), but a very poor low-pass filter (the action in the frequency domain) (Smith, 1997).

As can be seen, the "smoothing power" of this filter is dependent on the number of samples taken into account in the average or, in other words, on the extent *N*: the more the filter extent grows, the more the highest frequencies are dampened, and the more the high frequency noise is attenuated. Therein lies the interest of increasing *N*.

A useful definition of the cuttoff frequency could be the lowest frequency for which $|H_\omega| = 0$, i.e. for which $\frac{\omega N}{2} = \pi$. This gives the cuttoff frequency $f_{Hz} = \frac{1}{N\Delta t}$, where $N\Delta t$ is the extent of the filter.

First derivative backward difference (first order scheme)

$$|H_\omega| = \left| \frac{2}{\Delta T} sin\left(\frac{\omega d}{2} \right) \right|$$

The case $d = 1$ means that the differentiation step is chosen equal to the sample step (i.e. the step between two consecutive samples). This choice turns out to amplify the higher frequencies while absorbing the lower ones. This explains why this backward "one step" difference scheme is affected in a way so sensitive by the high frequency noise in the input signal: it causes it literally to explode.

On the other side, the more the differentiation step grows, the more the modulation equilibrates at the same level for all the represented frequencies, the more this amplification level decreases, and the more the global noise is attenuated. Therein lies the interest of increasing *d*.

Anyway, this difference scheme will never wipe off the higher frequencies of the input signal: the frequency range of the input will always remain the same in the output.

The maximum value is $|H_\omega|_{max} = \frac{2}{\Delta T}$, where $\Delta T = d\Delta t$. This means that doubling the differentiation step ΔT will double the attenuation of the noise magnitude.

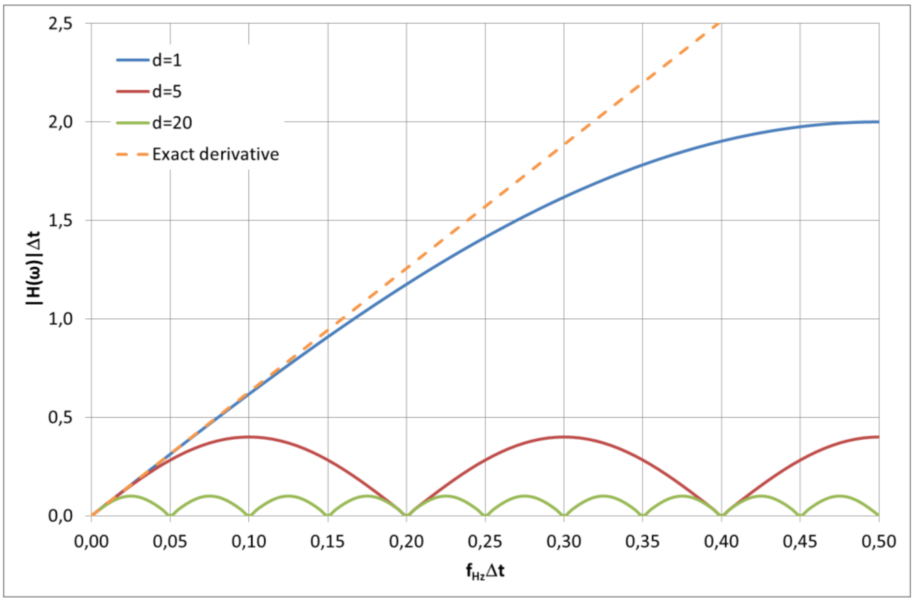


Figure 14 – First order backward difference with a d-differentiation step

First derivative backward difference (second order scheme)

$$|H_\omega| = \frac{1}{\Delta T}\sqrt{3\cos^2(\omega d) - 8\cos(\omega d) + 5}$$

The global shape of the curves and their relative proportions are the same as the ones for the backward first order scheme above; so does their interpretation.

The maximum value is $|H_\omega|_{max} = \frac{4}{\Delta T}$, where $\Delta T = d\Delta t$. This means that doubling the differentiation step ΔT will double the attenuation of the noise magnitude. But this maximum value is also twice the maximum value of the backward first order scheme above. This means that, for a same differentiation step ΔT, the backward first order scheme will attenuate the noise magnitude twice as much than the backward second order scheme (see Figure 9 for an example).

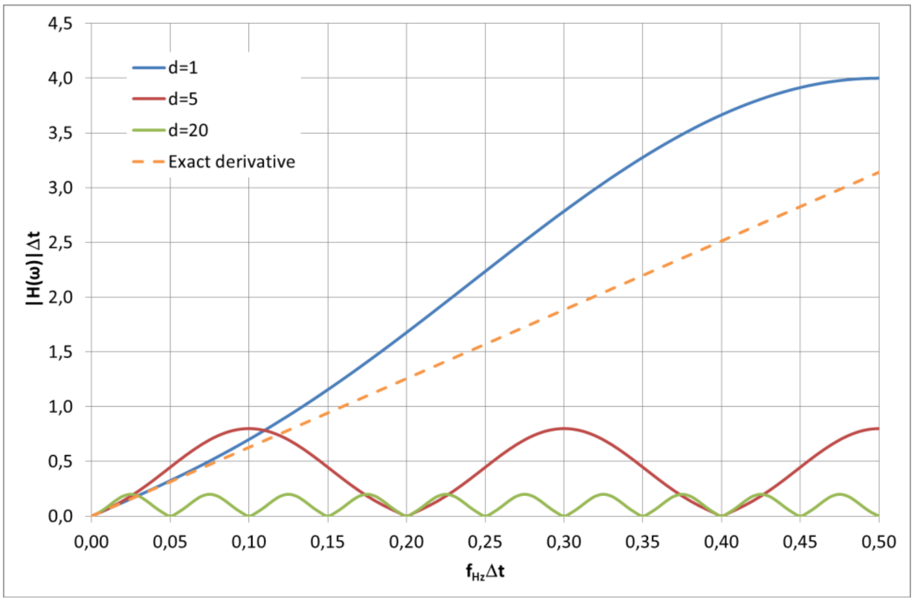


Figure 15 – Second order backward difference with a d-differentiation step

First derivative backward difference (third order scheme)

$$\left|H_{\omega}\right| = \frac{1}{3\Delta T}\sqrt{2(87 - 91\cos(\omega d) + 22\cos(\omega d))\sin^{2}\left(\frac{\omega d}{2}\right)}$$

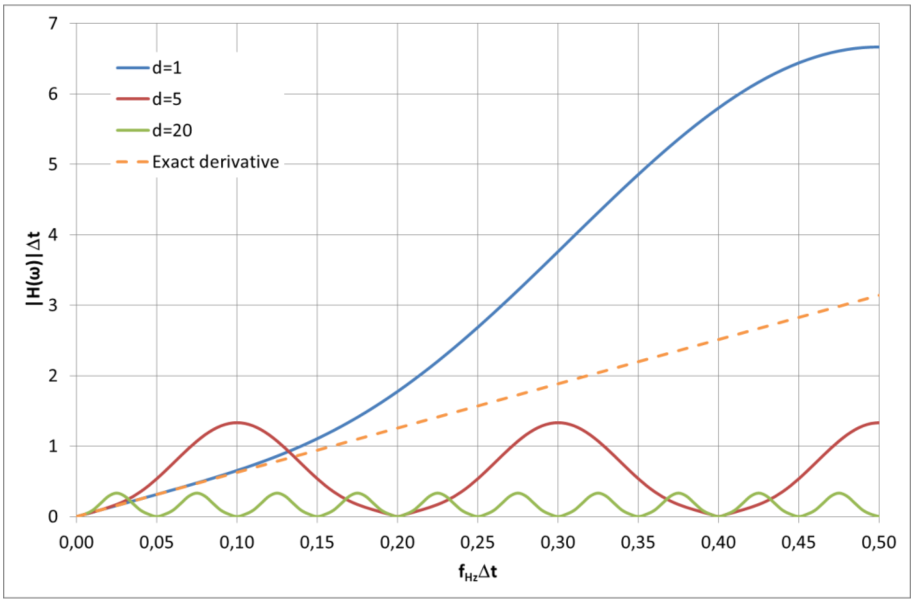


Figure 16 – Third order backward difference with a d-differentiation step

The global shape of the curves and their relative proportions are the same as the ones for the backward first and second orders scheme above; so does their interpretation.

The maximum value is $\left|H_{\omega}\right|_{max} = \frac{20}{3\Delta T}$, where $\Delta T = d\Delta t$. This means that doubling the differentiation step ΔT will double the attenuation of the noise magnitude. But this maximum value is also 3,33 times the maximum value of the backward first order scheme above. This means that, for a same differentiation step ΔT, the backward first order scheme will attenuate the noise magnitude 3,33 times as much than the backward third order scheme (see Figure 9 for an example).

First derivative centered difference (second order scheme)

$$|H_\omega| = \left| \frac{1}{\Delta T} \sin(\omega d) \right|$$

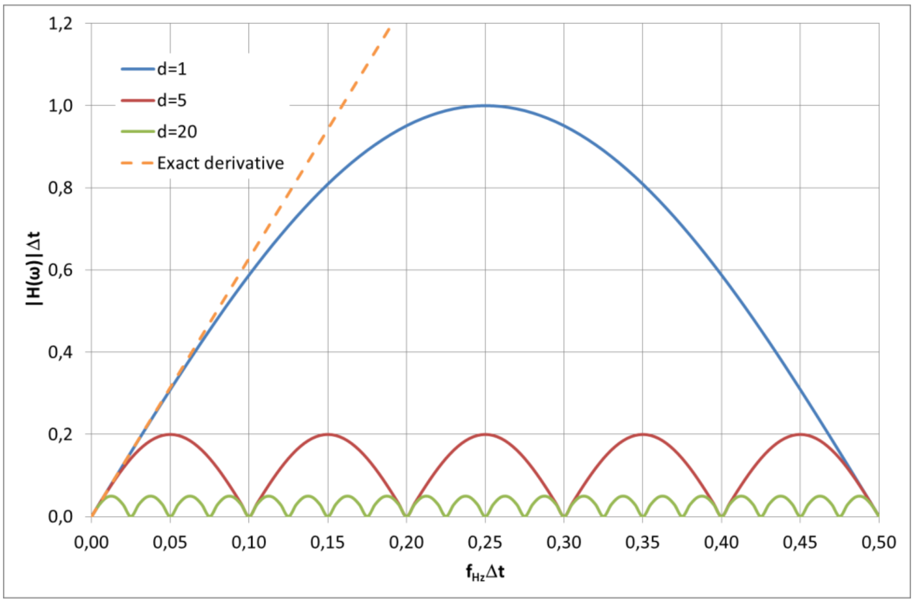


Figure 17 – Second order centered difference with a d-differentiation step

A direct observation is that $|H_\omega(d=2\beta)|_{backward\,1st\,order} = |H_\omega(d=\beta)|_{centered\,2nd\,order}$. The interpretation is thus the same as the one for the backward first order scheme, wherein the only even values of d remains, meaning that the worst case "$d = 1$" encountered in this last scheme is avoided here.

The maximum value is $|H_\omega|_{max} = \frac{1}{\Delta T}$, where $\Delta T = d\Delta t$. This means that doubling the differentiation step ΔT will double the attenuation of the noise magnitude. But this maximum value is also half the maximum value of the backward first order scheme above. This means that, for a same differentiation step ΔT, the centered second order scheme will attenuate the noise magnitude twice as much than the backward first order scheme.

Second derivative backward difference (first order scheme)

$$|H_\omega| = \frac{4}{\Delta T^2} \sin^2\left(\frac{\omega d}{2}\right)$$

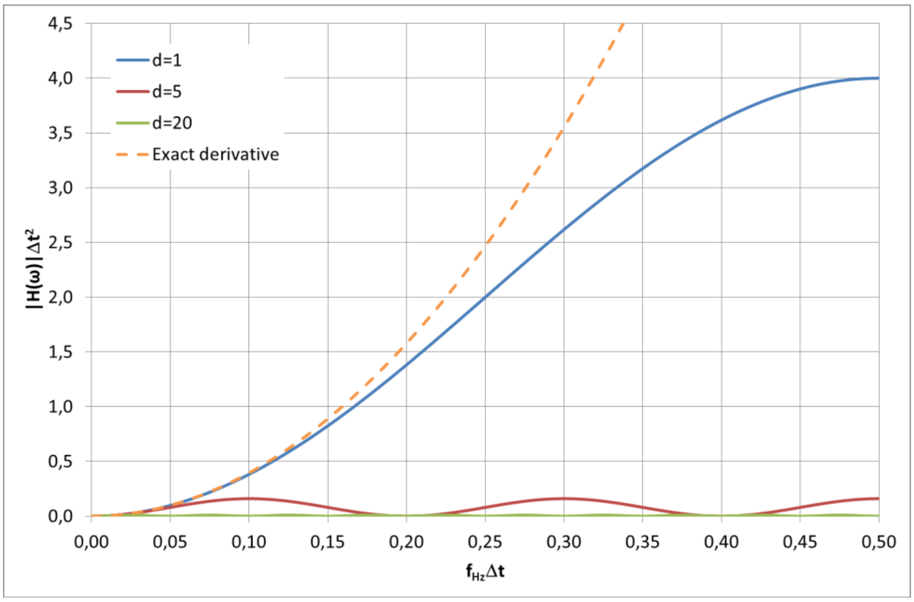


Figure 18 – First order backward difference with a d-differentiation step

The case $d = 1$ means that the differentiation step is chosen equal to the sample step (i.e. the step between two consecutive samples). This choice turns out to amplify the higher frequencies while absorbing the lower ones. This explains why this backward "one step" difference scheme is affected in a way so sensitive by the high frequency noise in the input signal: it causes it literally to explode.

On the other side, the more the differentiation step grows, the more the modulation equilibrates at the same level for all the represented frequencies, the more this amplification level decreases, and the more the global noise is attenuated. Therein lies the interest of increasing *d*.

Anyway, this difference scheme will never wipe off the higher frequencies of the input signal: the frequency range of the input will always remain the same in the output.

The maximum value is $\left|H_\omega\right|_{max} = \dfrac{4}{\Delta T^2}$, where $\Delta T = d\Delta t$. This means that doubling the differentiation step ΔT will quadruple the attenuation of the noise magnitude.

5.4.2 Phase spectrum

From the perspective of the Fourier representation as being a sum of sine and cosine waves, the phase spectrum defines the relative alignment of the waves. While this does not influence the energy distribution in frequency, this phase alignment does have a significant effect on the shape in the time domain (Prandoni & Vetterli, 2008). By its "shifting" action, the phase θ of an operator will delay or advance the input signal in the time domain, as a function of ω. This corresponds to the distortion and time shift effects of the operator.

Remembering

- the definition of the DTFT $X_\omega = \sum_{i=-\infty}^{\infty} x_i e^{-j\omega i}$,
- the frequency domain description of an LTI system $Y_\omega = X_\omega . H_\omega$, and
- the polar form of the frequency response $H_\omega = \left|H_\omega\right| e^{j\theta_\omega}$,

it follows

$$Y_\omega = |H_\omega| \sum_{i=-\infty}^{\infty} x_i e^{-j\omega(i+\tau_\varphi)}$$

The dimensionless parameter $\tau_\varphi(\omega) = -\frac{\theta_\omega}{\omega}$ is called the "phase delay" of the system, and is a function of frequency. Using the relation between dimensionless and dimensional angular frequencies $\omega i = \omega t_i / \Delta t = \omega_{Hz} t_i$, the argument of the wave components becomes $-\omega_{Hz}(t_i + \tau_\phi)$. It appears now clearly that the dimensional form of the phase delay $\tau_\phi(\omega) = -\frac{\theta_\omega}{\omega}\Delta t$ – where θ_ω is the phase of the frequency response of the system – gives the time delay experienced by each wave component of a signal through the system.

The interpretation is as follows:

- when the phase experiences negative values, the phase delay is positive, meaning that the operator delays the input signal (i.e. shift it to the right in the time domain),
- when the phase experiences positive values, the phase delay is negative, meaning that the operator advances the input signal (i.e. shift it to the left in the time domain).

Quite generally, the phase of the operator will be different for the various frequencies, so does the phase delay. This delay variation means that signals consisting of multiple frequency wave components will undergo distortion because these wave components are not shifted by the same amount of time at the output of the system. This phenomenon changes the shape of the output signal compared to the expected one. Sufficiently large delay variation can cause problems such as poor fidelity of the system (Pinki & Mehra, 2014).

In contrast, if the phase θ of the operator is linear (in ω) within its passband, then the phase will shift all the wave components of the input signal as a single block. Such linear phase operator is said to be "distortionless".

The present purpose is to determine the time shift resulting of the use of a time-discrete scheme instead of the related ideal system. The phase delay of the system is then the relevant parameter. One need only compare the phase delay of the scheme under study with the phase delay of the related ideal system. Two situations will be encountered here.

1. Filters that are only dedicated to noise elimination are not intended to change the basic shape of signals to which they apply. This feature is precisely one of the main ones that could be expected from a filter. In this first case, the output signal must simply be compared with the input signal, and the time shift produced by the system is given by the phase delay of the system.
2. Derivatives of signals, on the contrary, fall into the category of systems that change significantly the shape of signals to which they apply, and also their nature (this may in fact be the reason why the systems of this category are being designed and used). In this case, the only phase delay of the system describes a delay measurement that becomes somewhat arbitrary and meaningless. Comparing the output signal with the input signal doesn't make sense anymore. Finite difference schemes must be compared with their related ideal differentiator. Hence, the time shift produced by the system is:
   - $\Delta\tau_\phi = \tau_\phi^{Scheme} - \tau_\phi^{Ideal\,dif.} = \left(\frac{\pi}{2} - \theta_\omega^{Scheme}\right)\frac{\Delta t}{\omega}$ for first derivatives,
   - $\Delta\tau_\phi = \tau_\phi^{Scheme} - \tau_\phi^{Ideal\,dif.} = (\pi - \theta_\omega^{Scheme})\frac{\Delta t}{\omega}$ for second derivatives,

- …

In other words, the quantity $\Delta\tau_{\phi}(\omega)$ is nothing but the numerical time delay τ_i – defined in the above graphical approach – decomposed as a function of the frequency.

Unwrapping step

Before introducing the detailed phase spectra, the attention must be drawn on a subtle handling that must be performed when computing the phase. This processing is called "phase unwrapping" (Smith, 1997).

The phase has been defined as $\theta_{\omega} = arctan\left(\frac{\Im(H_{\omega})}{\Re(H_{\omega})}\right)$. Using any software to compute the arctangent will generate a chopped signal, as can be seen on Figure 19 (raw phase curve). The apparent discontinuities in the signal are a result of the computer algorithms picking their favorite choices from an infinite number of equivalent possibilities. The smallest possible value is always chosen, keeping the phase in the range $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Actually, there is no mathematic reason why the phase should be limited in this range. The understanding of the phase therefore requires first to remove these discontinuities, even if it means that the phase extends above $-\frac{\pi}{2}$ or below $\frac{\pi}{2}$. This unwrapping step shall be performed following the two successive handling:

1. if both the real and imaginary parts are negative, subtract π radians from the calculated phase; if the real part is negative and the imaginary part is positive, add π radians;
2. add or subtract integer multiples of 2π from each sample, where the integer is chosen to minimize the discontinuities between successive points, as can be seen on Figure 20.

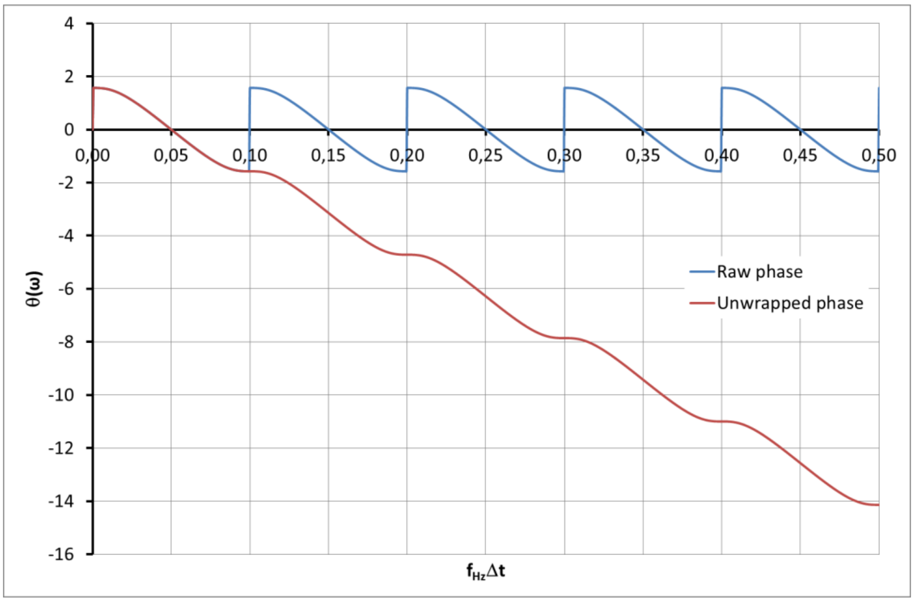


Figure 19 – Unwrapping the phase

APPLICATIONS

Backward moving average on N points

$$\theta=-\frac{\omega(N-1)}{2}\Rightarrow\tau_{\phi}=\frac{(N-1)}{2}\Delta t$$

The backward moving average on N points generates a positive delay independent of the frequency. The average calculated by this filter is thus shifted half of the extent of the filter to the right in the time domain. This means that doubling the extent of the filter N will double the delay. Since the time shift is independent of the frequency ω, no distortion is introduced.

First derivative backward difference (first order scheme)

$$\theta=-\frac{(\omega d-\pi)}{2}\Rightarrow\tau_{\phi}=\frac{(\omega d-\pi)}{2}\frac{\Delta t}{\omega}$$

$$\text{and } \Delta\tau_{\phi}=\frac{d}{2}\Delta t=\frac{\Delta T}{2}$$

where $\Delta T=d\Delta t$. The backward first order scheme generates a positive delay independent of the frequency. The derivative calculated by this difference scheme is thus shifted half of the differentiation step to the right in the time domain. This means that doubling the differentiation step ΔT will double the delay. Since the time shift is independent of the frequency ω, no distortion is introduced.

First derivative backward difference (second order scheme)

$$\theta=\arctan\left(\frac{\sin(\omega d)(2-\cos(\omega d))}{(1-\cos(\omega d))^{2}}\right)\Rightarrow\tau_{\phi}=-\frac{\Delta t}{\omega}\arctan\left(\frac{\sin(\omega d)(2-\cos(\omega d))}{(1-\cos(\omega d))^{2}}\right)$$

$$\text{and } \Delta\tau_{\phi}=\left(\frac{\pi}{2}-\arctan\left(\frac{\sin(\omega d)(2-\cos(\omega d))}{(1-\cos(\omega d))^{2}}\right)\right)\frac{\Delta t}{\omega}$$

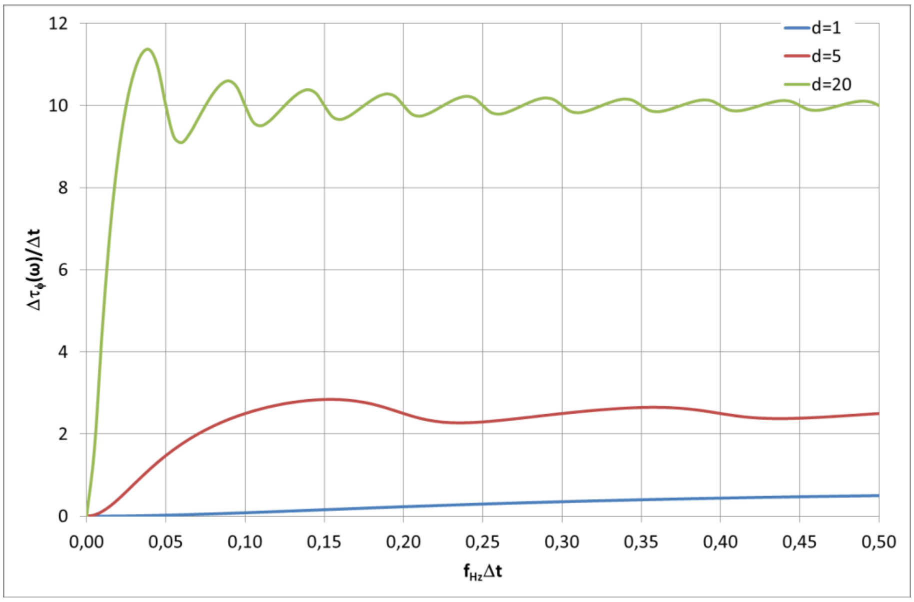


Figure 20 – Second order backward difference with a d-differentiation step

The backward second order scheme generates thus a positive delay dependent of the frequency. As a consequence, some distortion is expected. However, it appears that only the lowest frequencies suffer a substantial variable delay, and that the remaining ones – i.e. the greatest part of the frequency range – undergo a time delay of the same order of magnitude, equals to $\Delta\tau_\phi = \frac{d\Delta t}{2} = \frac{\Delta T}{2}$.
The more the differentiation step grows, the more this trend is noticeable. As the lowest frequencies depict the general appearance of the signal, while the highest ones depict the noise interference, this scheme is finally expected to produce a more moderate time-delay of the basic shape of the derivative than the first order scheme, at the expense of some distortion of this basic shape (see Figure 9).

First derivative backward difference (third order scheme)

$$\theta = \arctan\left(\frac{\sin(\omega d)(8-9\cos(\omega d)+4\cos^2(\omega d))}{1-6\cos(\omega d)+9\cos^2(\omega d)-4\cos^3(\omega d)}\right)$$

$$\Rightarrow \tau_\phi = -\frac{\Delta t}{\omega}\arctan\left(\frac{\sin(\omega d)(8-9\cos(\omega d)+4\cos^2(\omega d))}{1-6\cos(\omega d)+9\cos^2(\omega d)-4\cos^3(\omega d)}\right)$$

$$\text{and } \Delta\tau_\phi = \left(\frac{\pi}{2} - \arctan\left(\frac{\sin(\omega d)(8-9\cos(\omega d)+4\cos^2(\omega d))}{1-6\cos(\omega d)+9\cos^2(\omega d)-4\cos^3(\omega d)}\right)\right)\frac{\Delta t}{\omega}$$

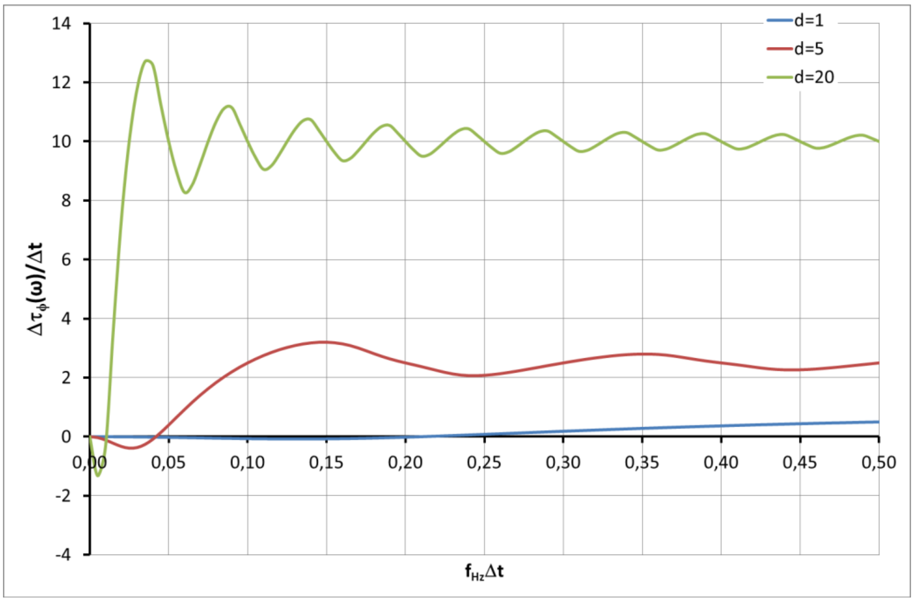


Figure 21 – Third order backward difference with a d-differentiation step

The global shape of the curves, their relative proportions and the asymptotic orders of magnitude are the same as the ones for the backward second order scheme above; so does their interpretation (see Figure 9).

A important difference appears though in the lowest frequencies. Although these ones remain weakly shifted, this shift now turns on to be negative, meaning a time-advanced effect! Once again, this can be perceived on Figure 9. From the expression of $\Delta\tau_\phi$ above, one finds that these negatives values are encountered from 0 Hz up to $\omega d = A\cos\left(\frac{1}{4}\right)$ and thus $f_{Hz}\Delta t = \frac{1}{2\pi d}A\cos\left(\frac{1}{4}\right) \approx \frac{0{,}21}{d}$.

First derivative centered difference (second order scheme)

$$\theta = \frac{\pi}{2} \Rightarrow \tau_\phi = -\frac{\pi}{2}\frac{\Delta t}{\omega}$$
$$\text{and } \Delta\tau_\phi = 0$$

The centered second order scheme generates thus neither delay, nor distortion.

Second derivative backward difference (first order scheme)

$$\theta = -(\omega d - \pi) \Rightarrow \tau_\phi = (\omega d - \pi)\frac{\Delta t}{\omega}$$
$$\text{and } \Delta\tau_\phi = d\Delta t = \Delta T$$

where $\Delta T = d\Delta t$. The backward first order scheme generates a positive delay independent of the frequency. The derivative calculated by this difference scheme is thus shifted of the differentiation step to the right in the time domain. This means that doubling the differentiation step ΔT will double the delay. Since the time shift is independent of the frequency ω, no distortion is introduced.

6 REVIEW OF THE CURRENT SITUATION

Basic finite difference schemes on the one hand, and the moving average filter on the other, have been defined and then analysed. Regarding the first ones, it appears that:

- the more the differentiation step grows:
  - the more the noise is attenuated and equilibrated on the whole range of the represented frequencies,
  - the more the time shift increases,
- the more the order of the scheme grows:
  - the less the noise is attenuated,
  - the more the time shift is attenuated.

This is – as always – a question of compromises.

From this point, choices will be made to progress towards concrete solutions. These choices will be based on following criteria, and are led by the need for a practical, understandable and easily implementable field method.

1. **Noise dampening**

   The primary objective remains to dampen the high frequency noise from the time signal under processing. According to the magnitude spectrum analyses above, too small differentiation step will be discarded, as well as schemes with orders greater than 2 (see Figure 9 for an explicit example). While these choices will increase the time-shift, this one may however be kept under control thanks to further choices (see below).

2. **Real-time processing**

   Only causal schemes will be considered, meaning only backward schemes.

3. **Time-shift correction**

   According to the phase spectrum analyses above, amongst the backward schemes, only the first order ones can be corrected in a perfect way, because of their frequency independent delay. The correction consists of simply time-advancing the result by $\frac{\Delta T}{2}$ for the first derivative, ΔT for the second derivative, …

4. **Real-time results**

   Ideally, a strictly real-time method would be welcome. According to the phase spectrum analyses above, this is not strictly feasible with backward schemes. Depending on the importance or not of the immediate availability of results, two possibilities remain at disposal.

   - Immediate result is needed
     In this first case, the second order scheme could be chosen, supplying a moderate delayed result, whose time-shift cannot be corrected.
   - Slightly delayed result is acceptable
     In this second case – and in all cases where a small delay can be accepted before the results become available – the first order scheme will be chosen, supplying a result delayed of a

well-known single value (see above). The corrected result is then known with a lag of $\frac{\Delta T}{2}$ for the first derivative, ΔT for the second derivative, …

The last case should be preferred whenever possible.

5. Fidelity

According to the phase spectrum analyses above, amongst the backward schemes, only the first order ones are distorsionless, because of their linear (in ω) phase.

6. Simplicity

Of course, easily implementable schemes are privileged. That's why, once again, first order schemes should be chosen whenever possible. They are processed with only 2 signal samples (1st derivative) instead of 3 for the second order. This means also that they don't need to make use of too old values, just the most recently acquired ones.

The first order backward schemes are thus the most suitable when the objectives are the ones stated here above. The second order backward schemes should only be used in case where immediate results are needed.

For all these reasons, from this point, only the first order backward schemes will be considered. The next section will now focus on how these schemes can be simply improved.

7 COMBINED SCHEMES

One could read in the literature that denoising the data before or after differentiating does not generally give satisfactory results. Although such denoising step doesn't indeed solve all the problems, it would be wrong to put this tool aside definitely. The way it improves the outcome is significant compared to its simplicity. And simplicity is precisely what is privileged in this document.

The idea is now to combine sequentially the moving average filter and the basic first order backward schemes to boost both noise reduction and fidelity. The resulting scheme will be named *combined scheme*.

It has been shown that LTI systems are fully described in the time domain by the convolution sum $y_i = x_i * h_i$, or in the frequency domain by the product $Y_\omega = X_\omega . H_\omega$. Denoting by the superscripts *Comb*, *MA* and *Scheme* the resulting combined scheme, the moving average filter, and the basic finite difference scheme, it follows in the time domain $h_i^{Comb} = h_i^{MA} * h_i^{Scheme}$, and in the frequency domain $H_i^{Comb} = H_i^{MA} . H_i^{Scheme}$. Due to the commutative property of the convolution sum and of the multiplication, whatever the sequence of application "difference scheme – moving average" or "moving average – difference scheme", the result will be the same.

7.1 FIRST DERIVATIVE COMBINED SCHEME

Instead of performing a first order backward difference scheme with a d-differentiation step, the investigated solution will be to perform successively a first order m-differentiation step scheme on a $n+1$-extent backward moving average, where $m+n=d$. This means that the latter two are computed on extents of $m+1$ and $n+1$ samples respectively, and thus on time-extents of $m\Delta t$ and $n\Delta t$. The resulting combined scheme is meanwhile computed on a total extent of $d+1$ samples, and thus on a total time-extent of $d\Delta t$. The idea is therefore to use data issued from the same past time sequence.

In the time domain

$$y_i = \frac{\sum_{k=0}^{n} x_{i-k} - \sum_{k=0}^{n} x_{i-m-k}}{(n+1)\Delta T}$$

where $\Delta T = m\Delta t$, and in the frequency domain

$$Y_\omega = X_\omega . H_\omega^{Scheme} . H_\omega^{MA} = X_\omega . \left| \frac{2}{(n+1)\Delta T} \frac{\sin\left(\frac{\omega m}{2}\right)\sin\left(\frac{\omega(n+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right| e^{-j\left(\omega\frac{m+n}{2}-\frac{\pi}{2}\right)}$$

In the following, only the case $m = n = \frac{d}{2}$ is deepened since it appears to be the most satisfactory. Hence, the frequency response of the combined scheme becomes

$$H_\omega^{Comb} = \left| \frac{2}{(m+1)\Delta T} \frac{\sin\left(\frac{\omega m}{2}\right)\sin\left(\frac{\omega(m+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right| e^{-j\left(\omega m-\frac{\pi}{2}\right)}$$

Magnitude spectrum

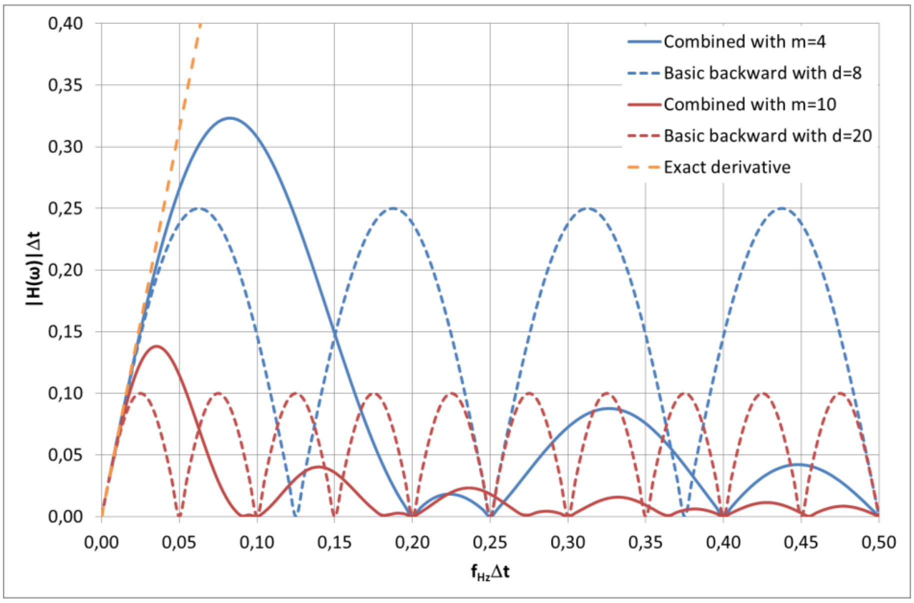


Figure 22 – Magnitude spectrum: comparison of the combined scheme with the basic first order backward difference, both having the same total scheme-extent

Contrary to the basic scheme (see Figure 14, where the modulation equilibrates the spectrum at the same level for all the represented frequencies), the combined scheme attenuates the highest frequencies. A useful definition of the cuttoff frequency could be the lowest frequency for which $|H_\omega| = 0$, i.e. for which $\frac{\omega(m+1)}{2} = \pi$. This gives the cuttoff frequency $f_{Hz} = \frac{1}{(m+1)\Delta t}$.
Moreover, the combined scheme attenuates the low frequencies less that the basic scheme. As the lowest frequencies depict the general appearance of the signal, the combined scheme is expected to represent the basic shape with a highest fidelity. Figure 23 and Figure 24 indeed illustrate this benefit: the broadening and flattening of the narrow features is now less pronounced.

Phase spectrum

$$\theta = -\left(\omega m - \frac{\pi}{2}\right) \Rightarrow \tau_\phi = \left(\omega m - \frac{\pi}{2}\right)\frac{\Delta t}{\omega}$$

and $\Delta\tau_\phi = m\Delta t = \frac{d}{2}\Delta t$

The combined scheme generates a positive delay independent of the frequency. The derivative calculated by this combined scheme is thus shifted half of the total extent to the right in the time domain. This means that doubling m – and thus d – will double the delay. Since the time shift is independent of the frequency ω, no distortion is introduced.

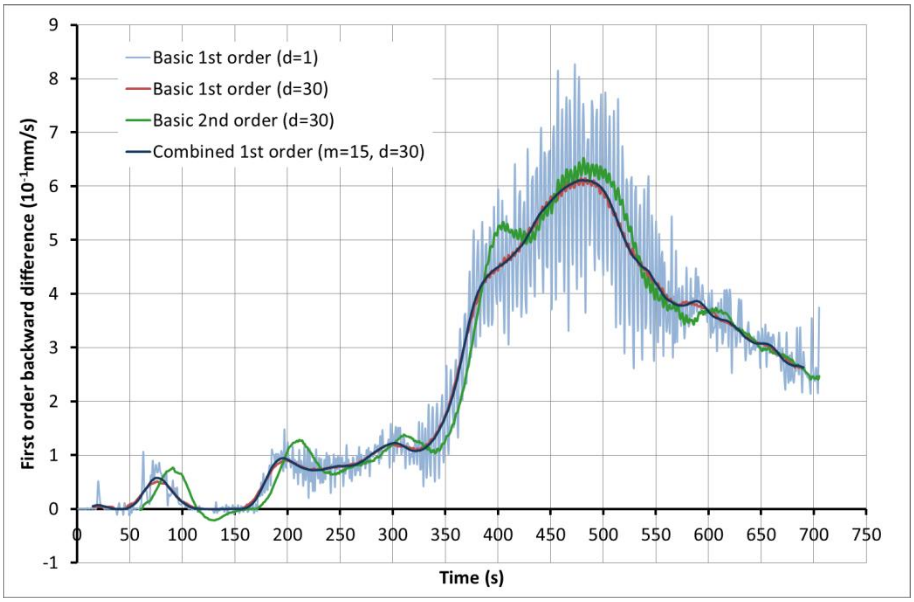


Figure 23 – First order basic scheme compared with its corresponding combined scheme, for a total scheme-extent of 30 seconds, with their 15 seconds time-shift correction (second order scheme is superimposed for information, with no possible time-shift correction)

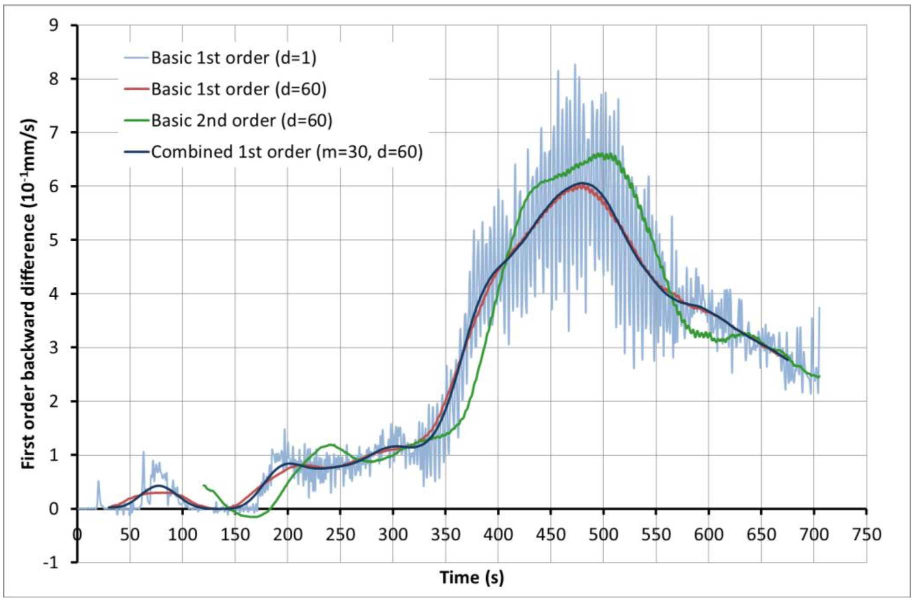


Figure 24 – First order basic scheme compared with its corresponding combined scheme, for a total scheme-extent of 60 seconds, with their 30 seconds time-shift correction (second order scheme is superimposed for information, with no possible time-shift correction)

Implementation

The proposed combined scheme is very easy to implement.
In a spreadsheet like Excel, assuming that the sampled time lies in the column A and the original sampled signal lies in the column B, then the basic scheme is computed by a formula like

...
=(B35-B5)/(A35-A5)
...

while the corresponding combined scheme is compute by a formula like

...

=(AVERAGE(B20:B35)-AVERAGE(B5:B20))/(A35-A20)

...

Nothing very onerous or complicated!

7.2 SECOND DERIVATIVE COMBINED SCHEME

Instead of performing a first order backward difference scheme with a d-differentiation step, the investigated solution will be to perform successively a first order *m*-differentiation step scheme on a $m+1$-extent backward moving average, where $m=\frac{2d}{3}$. This means that the latter two are computed on extents of $2m+1$ and $m+1$ samples respectively, and thus on time-extents of $2m\Delta t$ and $m\Delta t$. The resulting combined scheme is meanwhile computed on a total extent of $2d+1$ samples, and thus on a total time-extent of $2d\Delta t$. The idea is therefore to use data issued from the same past time sequence.

In the time domain

$$y_i = \frac{\sum_{k=0}^{m} x_{i-k} - 2\sum_{k=0}^{m} x_{i-m-k} + \sum_{k=0}^{m} x_{i-2m-k}}{(m+1)\Delta T^2}$$

where $\Delta T = m\Delta t$, and in the frequency domain

$$Y_\omega = X_\omega . H_\omega^{Scheme} . H_\omega^{MA} = X_\omega . \left| \frac{4}{(m+1)\Delta T^2} \frac{\sin^2\left(\frac{\omega m}{2}\right)\sin\left(\frac{\omega(m+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right| e^{-j\left(\omega\frac{3m}{2}-\pi\right)}$$

Magnitude spectrum

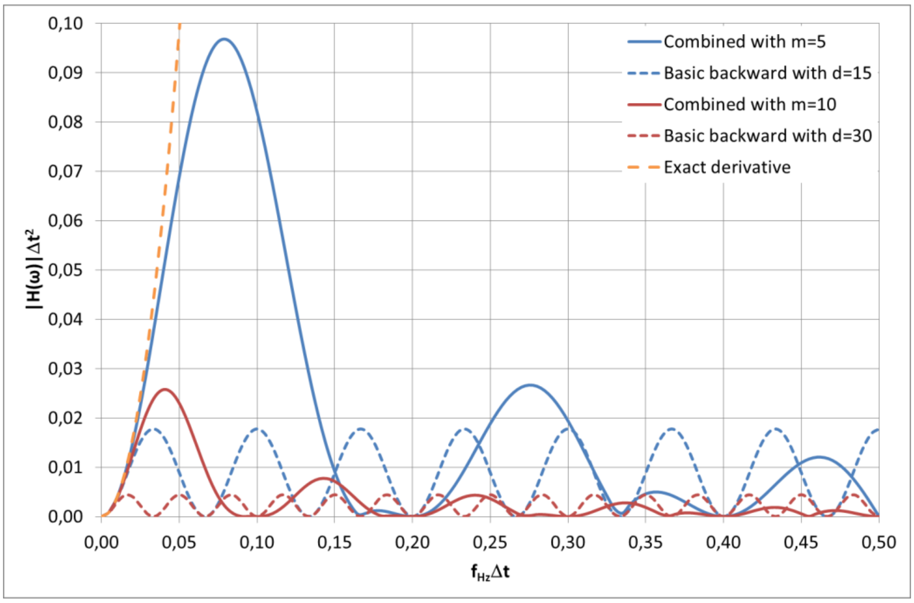


Figure 25 – Magnitude spectrum: comparison of the combined scheme with the basic first order backward difference, both having the same total scheme-extent

Contrary to the basic scheme (see Figure 18, where the modulation equilibrates the spectrum at the same level for all the represented frequencies), the combined scheme attenuates the highest frequencies. A useful definition of the cuttoff frequency could be the lowest frequency for which $|H_\omega| = 0$, i.e. for which $\frac{\omega(m+1)}{2} = \pi$. This gives the cuttoff frequency $f_{Hz} = \frac{1}{(m+1)\Delta t}$.
Moreover, the combined scheme attenuates the low frequencies much less that the basic scheme. As the lowest frequencies depict the general appearance of the signal, the combined scheme is expected to represent the basic shape with a highest fidelity.

Phase spectrum

$$\theta = -\left(\omega \frac{3m}{2} - \pi\right) \Rightarrow \tau_\phi = \left(\omega \frac{3m}{2} - \pi\right)\frac{\Delta t}{\omega}$$

$$\text{and } \Delta\tau_\phi = \frac{3m}{2}\Delta t = d\Delta t$$

The combined scheme generates a positive delay independent of the frequency. The derivative calculated by this combined scheme is thus shifted of the total extent to the right in the time domain. This means that doubling m – and thus d – will double the delay. Since the time shift is independent of the frequency ω, no distortion is introduced.

7.3 CONCLUSION

Compared to the basic scheme, the combined scheme has thus the advantage of better dampening the high frequency noise and better depicting the general appearance of the signal, while keeping the same delay behaviour and being processed from the same total extent of samples.

8 INFLUENCE OF THE SAMPLING PERIOD

At some point, the question of the sampling period Δt inevitably arises. This one can easily be answered through the spectrum analyses led above by noticing that – from the perspective of the analysis – increasing Δt is equivalent to increasing the differentiation step *d* while keeping Δt constant.

Firstly, the magnitude spectrum analysis (Figure 14) allows concluding that the more the sample step Δt (time step) grows, the more the modulation equilibrates at the same level for all the represented frequencies, the more this amplification level decreases, and the more the global noise is attenuated. Doubling the sample step Δt doubles the attenuation of the noise magnitude.

Secondly, the phase spectrum analysis allows concluding that the derivative is shifted half of the sample step Δt (time step) to the right in the time domain. Doubling the sample step Δt doubles the delay.

Last but not least, an obvious consequence goes along with an increase of the sample step Δt (time step): the data is acquired at the only sampled times $t_i = i\Delta t$, meaning that one has to wait Δt before getting the next measured data, and thus being able to compute the next value of the derivative. During each time interval Δt, no new information is available.

These properties are illustrated on the Figure 26, where only the round markers depict the available information, while the dotted intervals between the markers depict a lack of information. The combined scheme has just been drawn with a continuous line because it generates one data per second, and using some round markers would have given rise to a too dense frame of markers.

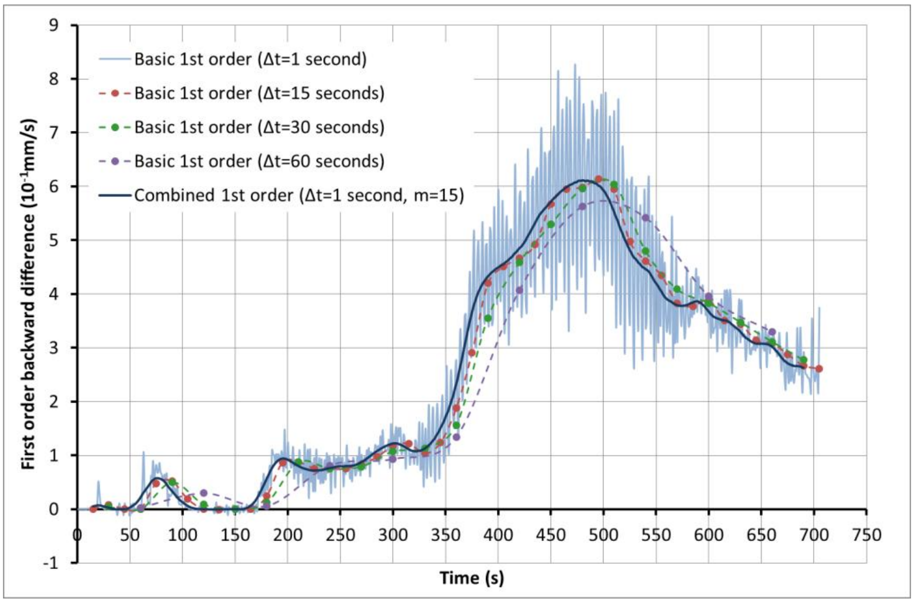


Figure 26 – Sampling period: comparison of the combined scheme with basic first order backward differences, all having the same total scheme-extent

9 ADVANCED METHOD

In this document, the choice was intentionally made to focus on methods as simple as possible based on basic finite-differences. In severe cases, this simplicity will prove to be an unsatisfactory compromise.

One will find in the literature various advanced methods to compute the filtered derivatives of noisy signals. These methods achieve this goal with more or less success depending on the encountered cases. However, these advanced methods will always require much more onerous computational resources and/or mathematical knowledge, while sometimes proving efficient in only specific situations. Here are some examples.

Simple approaches

Some methods involve smoothing basic handlings, like the least squares polynomial approximation or the smoothing spline (Knowles & Renka, 2014).

The Lanczos method transforms the problem into the calculation of the limit of an integral (Washburn, 2006). The Savitzky-Golay Smoothing Filters rely on local least-squares polynomial approximation (Luo, Ying, & Bai, 2005). Both are essentially centered method, and thus non-causal.

Holoborodko differentiators

Holoborodko proposes an original smooth noise-robust differentiators family (Holoborodko, 2008). Reminding that finite differences backward schemes can be written as

$$f_i^{(1)} \approx \frac{1}{\Delta t} \sum_{k=0}^{N} c_k f_{i-k}$$

the principle is to select the coefficients c_k such that the frequency response of the scheme H_ω will be as close as possible to the response of an ideal differentiator $H_\omega^{Ideal\,dif.} = j\omega$ in the low frequency region and smoothly tend to zero towards highest frequencies $\omega = \pi$. The chosen way to do this is to force H_ω to have high tangency order with $H_\omega^{Ideal\,dif.}$ at $\omega = 0$ as well as high tangency order with ω axis at $\omega = \pi$. This leads to a system of linear equations against c_k.
The tangency of H_ω with response of ideal differentiator at $\omega = 0$ is equivalent to exactness on monomials up to corresponding degree: $1, x, x^2, \ldots, x^n$. Choosing the simplest case $n = 1$ means forcing H_ω to be tangent at the 1st order at $\omega = 0$ with H_ω, which supplies the formula

$$c_k = \begin{cases} 1 & k = 0 \\ \dfrac{(N-1)!}{k!(N-k-1)!} \dfrac{N-2k}{N-k} & 0 \prec k \prec N \\ -1 & k = N \end{cases}$$

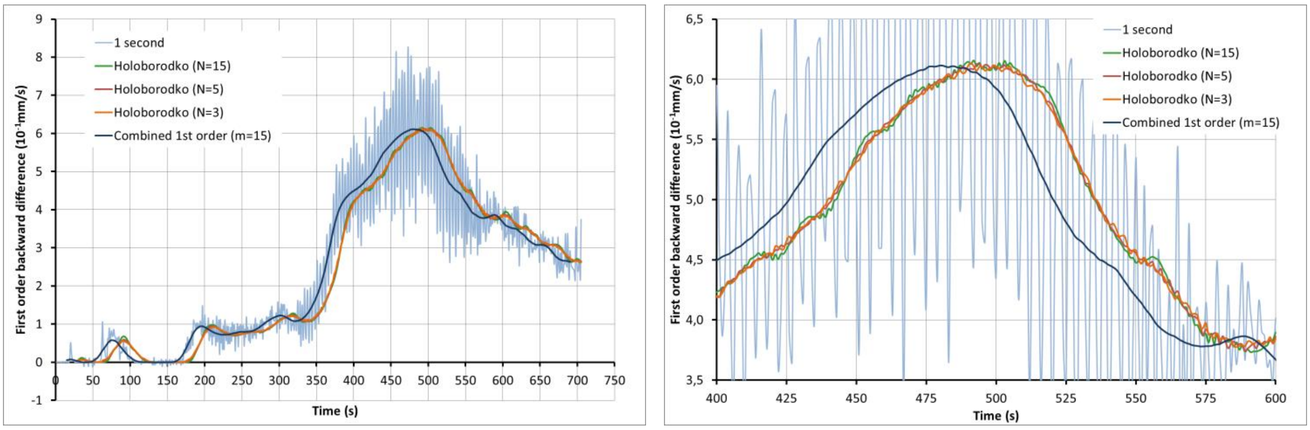
where $!$ refers to the factorial function. Especially, it appears that $c_{N-k} = -c_k$. Note that the case $N = 1$ reduces to $f_i^{(1)} \approx \frac{1}{2\Delta t}(f_i - f_{i-2})$, i.e. the basic backward 1st order scheme.
The Figures 27 below compares the Holoborodko differentiator for some values of N with the combined scheme:

$$N = 3: \; f_i^{(1)} \approx \frac{1}{4\Delta t}(f_i + f_{i-1} - f_{i-2} + f_{i-3})$$

$$N = 5:\ f_i^{(1)} \approx \frac{1}{16\Delta t}(f_i + 3f_{i-1} + 2f_{i-2} - 2f_{i-3} - 3f_{i-4} - f_{i-5})$$

$$N = 15:\ f_i^{(1)} \approx \frac{1}{16384\,\Delta t}(f_i + 13f_{i-1} + 77f_{i-2} + 273f_{i-3} + 673f_{i-4} + 1001f_{i-5} + 1001f_{i-6} + 429f_{i-7}$$
$$- 429f_{i-8} - 1001f_{i-9} - 1001f_{i-10} - 637f_{i-11} - 273f_{i-12} - 77f_{i-13} - 13f_{i-14} - f_{i-15})$$

For the comparison, these are all computed on a same total extent of 30 samples. Subjecting the Holoborodko schemes to the same time-shift correction would show how there are close to the combined scheme.



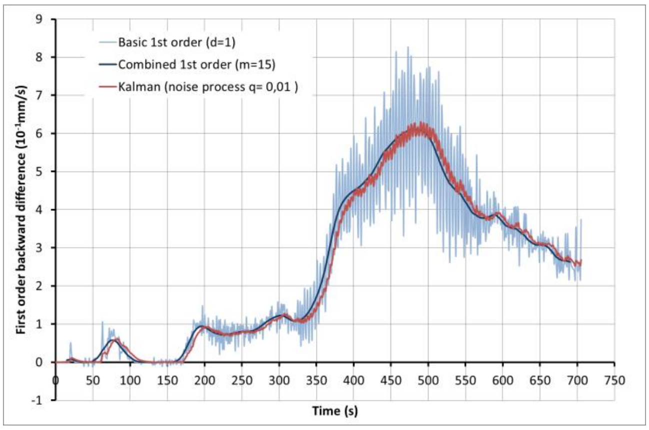
Figures 27 – Comparison of the Holoborodko differentiator with the combined scheme
(the combined scheme has been corrected from its 15 seconds time-shift delay)

<u>Total-variation regularization</u>

More subtle is the Tikhonov regularization and its enhanced form called “total-variation regularization” by Rick Chartrand (Chartrand, Numerical Differentiation of Noisy, Nonsmooth Data, 2011). The latter proposes an algorithm that accurately differentiates strongly noisy signals, including those which have a discontinuous derivative. The principle of this method consists of rewriting the derivative of a function as being the solution to the minimization of a given functional. Solving this problem then requires the iterative solving of a matrix equation involving the Hessian of the functional. This method is carried out on the whole data and is thus non-causal.

<u>Kalman filters</u>

Finally, the Kalman filter is a recursive Bayesian filter, especially optimal if the noise is Gaussian. This technique covers actually a wide range of applications, what makes its popularity. However, its efficiency relies on the availability of a fairly representative model of the state-space process in presence. The forming of such model requires knowing somewhat of the physic underlying the observed phenomenon.
Otherwise can the Kalman method only be based on a simplified general model. Quite generally, this will be the case in presence of noisy values whose measurement process cannot be formally linked to a known descriptive law. The case when the model parameters are time-invariant and the noise processes are stationary is then a convenient one (Einicke, 2012). Anyway, the Figure 28 shows that even such advanced method turns out to suffer from the same defects than basic schemes: the good old compromise between filtering and fidelity.



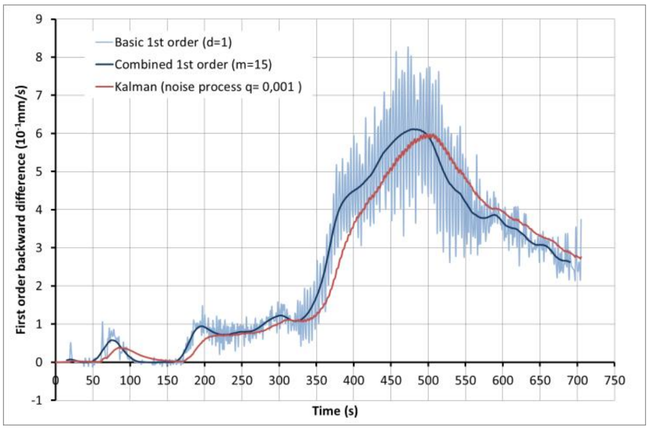


Figure 28 – Comparison of the Kalman filter with the combined scheme
(the combined scheme has been corrected from its 15 seconds time-shift delay)

This last illustration shows that advanced methods do not necessarily bring more efficient solution than a general and basic scheme can do.

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ANNEX 1 – MOVING MEDIAN FILTER

The moving average filter proves to be rather inappropriate when applied to signals noised by sharp peaks. This example given in Figure 29 below will be used to shortly introduce the concept of moving median filter.

In statistics, the median is the middle number in a given sequence of numbers, separating the higher half of a data sample from the lower half. The median of a finite list of numbers can be found by arranging all the observations from lowest value to highest value and picking the middle one.

All the interest of the median lies here: as long as no more than half of the data is contaminated by extreme values, the median will not give an arbitrarily extreme result. This essential property makes the median a robust statistical estimator of the central tendency, while the mean (average) is not (the latter is much more sensitive to outliers) (Liberty, 2010).

In a spreadsheet like Excel, just use the function "=MEDIAN(*range of cells*)".

The Figures 29a, b and c exemplify the efficiency of the moving median filter on a signal noised by sharp peaks, and allow apprehending the condition stated above "as long as no more than half of the data is contaminated by extreme values".

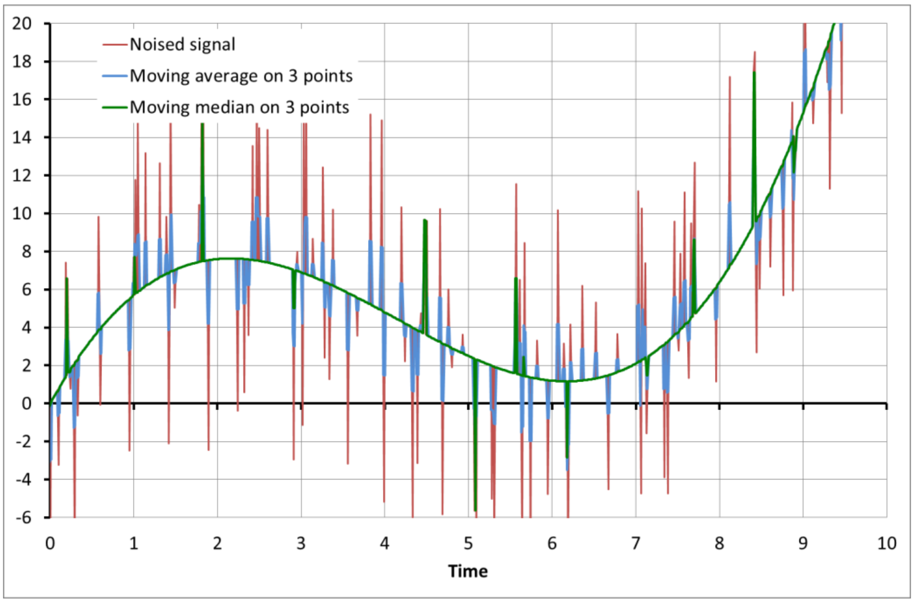


Figure 29a – Comparison of the moving average filter and the moving median filter carried out on 3 points (centered schemes)

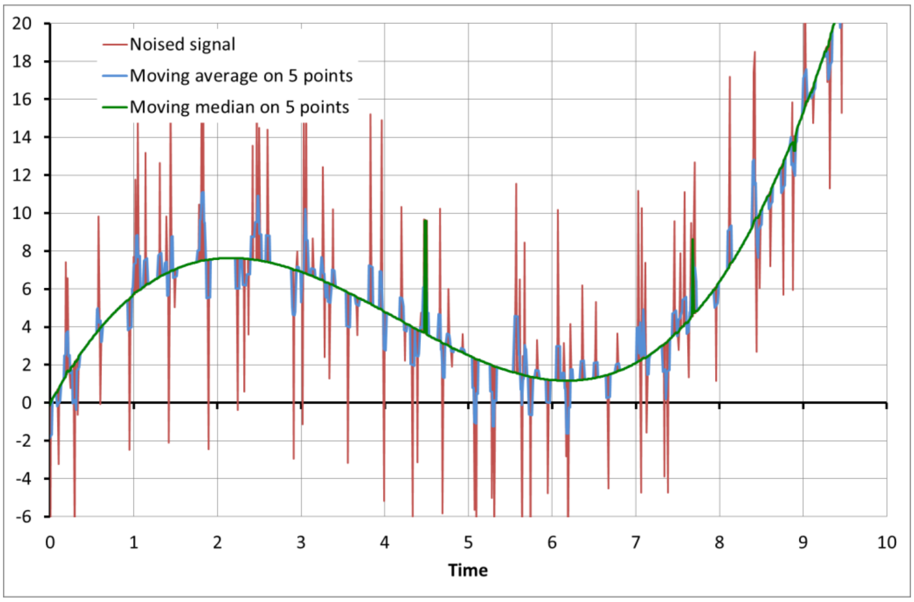


Figure 29b – Comparison of the moving average filter and the moving median filter carried out on 5 points (centered schemes)

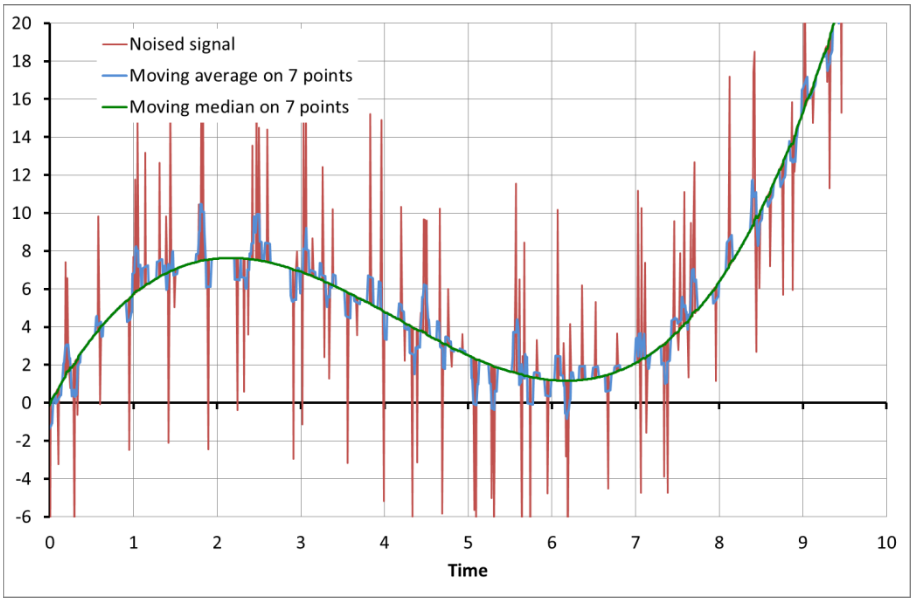


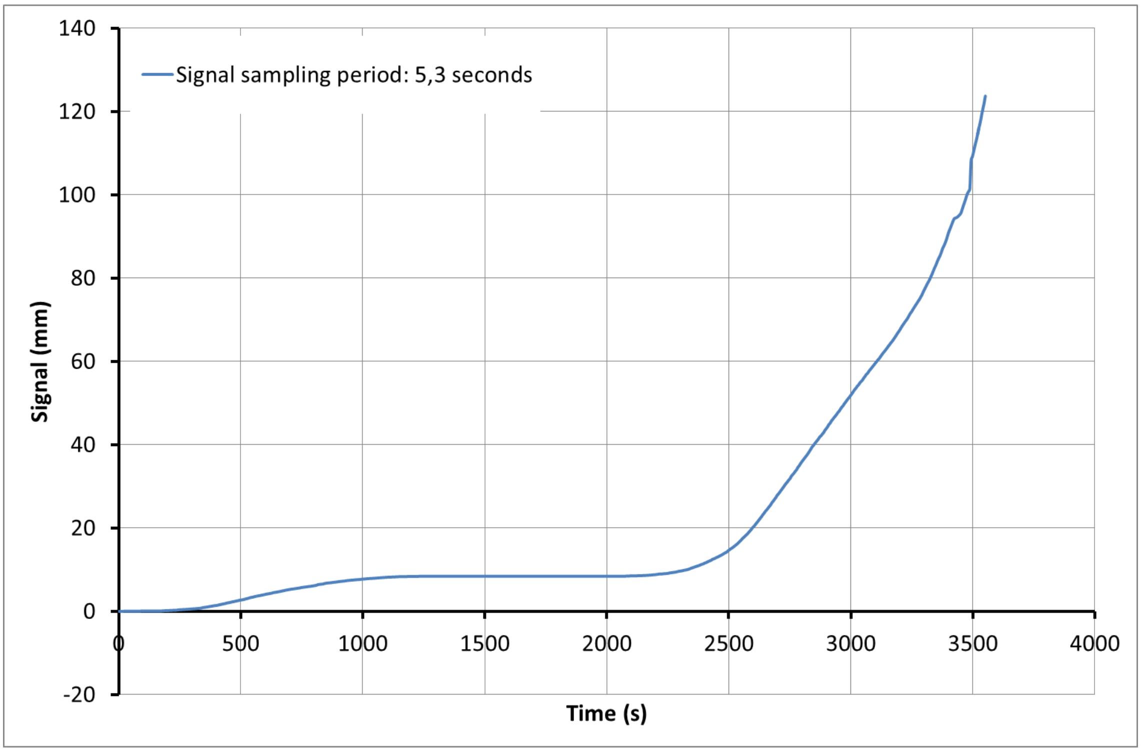
Figure 29c – Comparison of the moving average filter and the moving median filter carried out on 7 points (centered schemes)

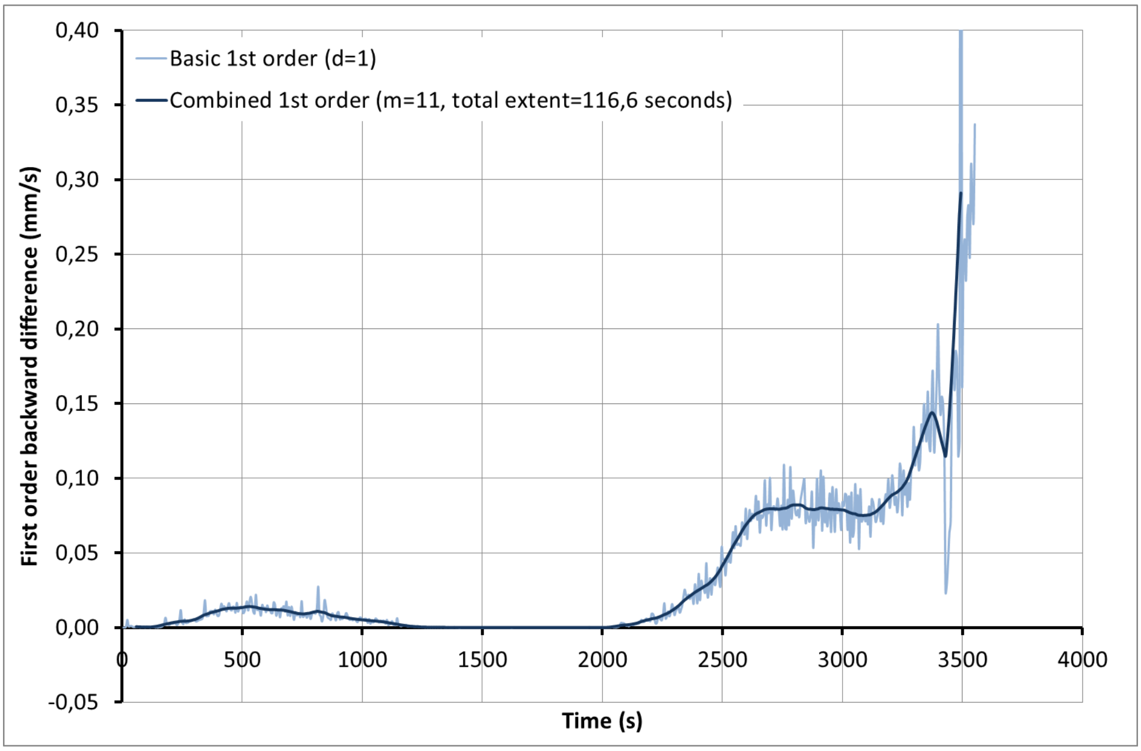
ANNEX 2 – APPLICATIONS

Below are given some illustrations based on real experimental data.
The parameters values of the combined 1st order scheme have not been chosen to answer to some well-defined filtering purpose. The resulting curves below thus don't claim to provide the best filtered approximation of the exact derivative, they simply depict what resulting effect is produced by the given values. Every user will should set the parameters values according to its own goals.

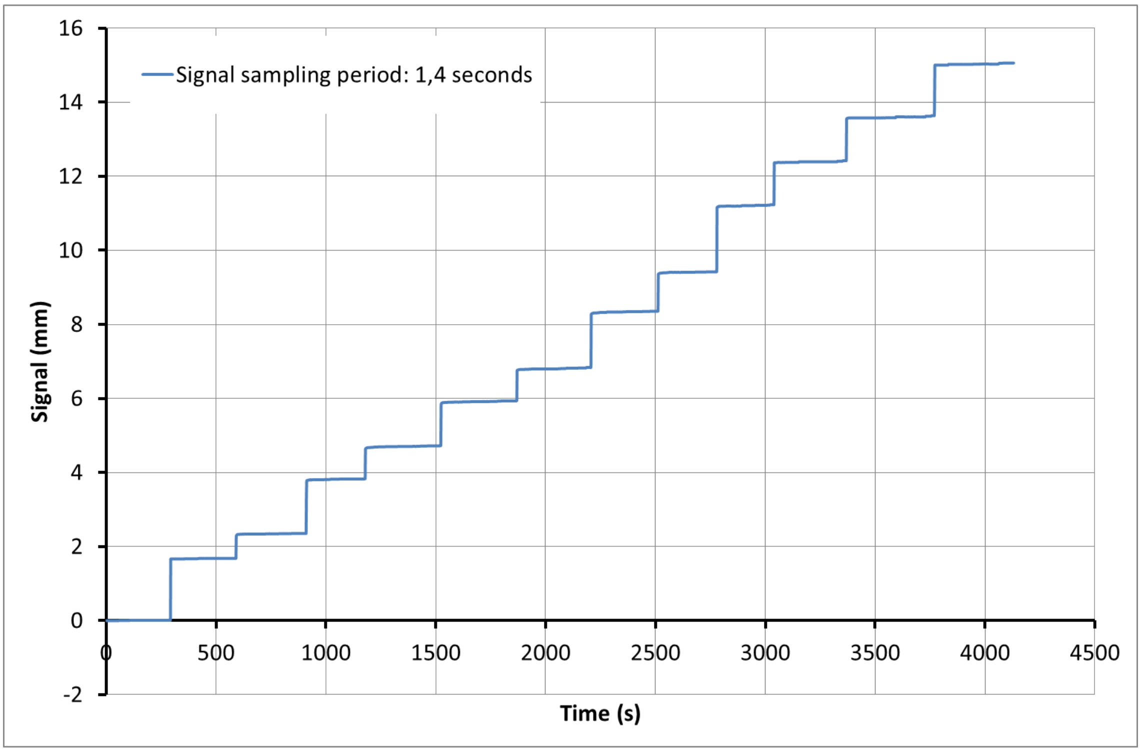
Note that the combined schemes below have been corrected from their time-shift delay.

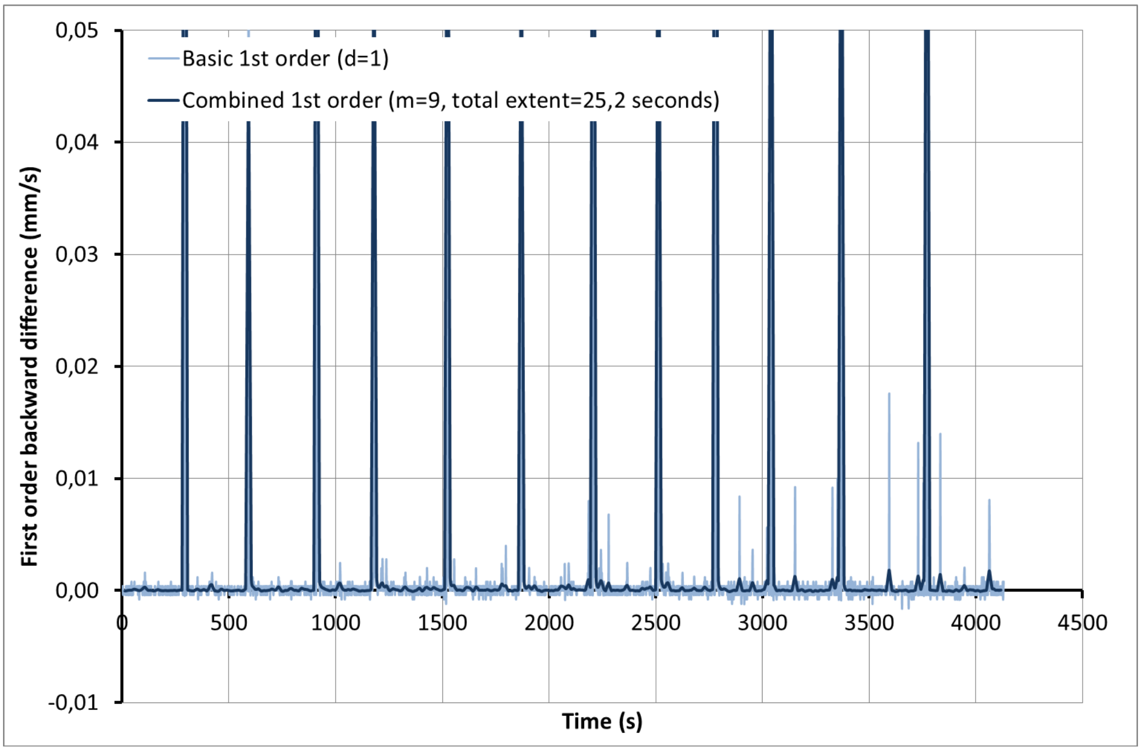
EXAMPLE 1

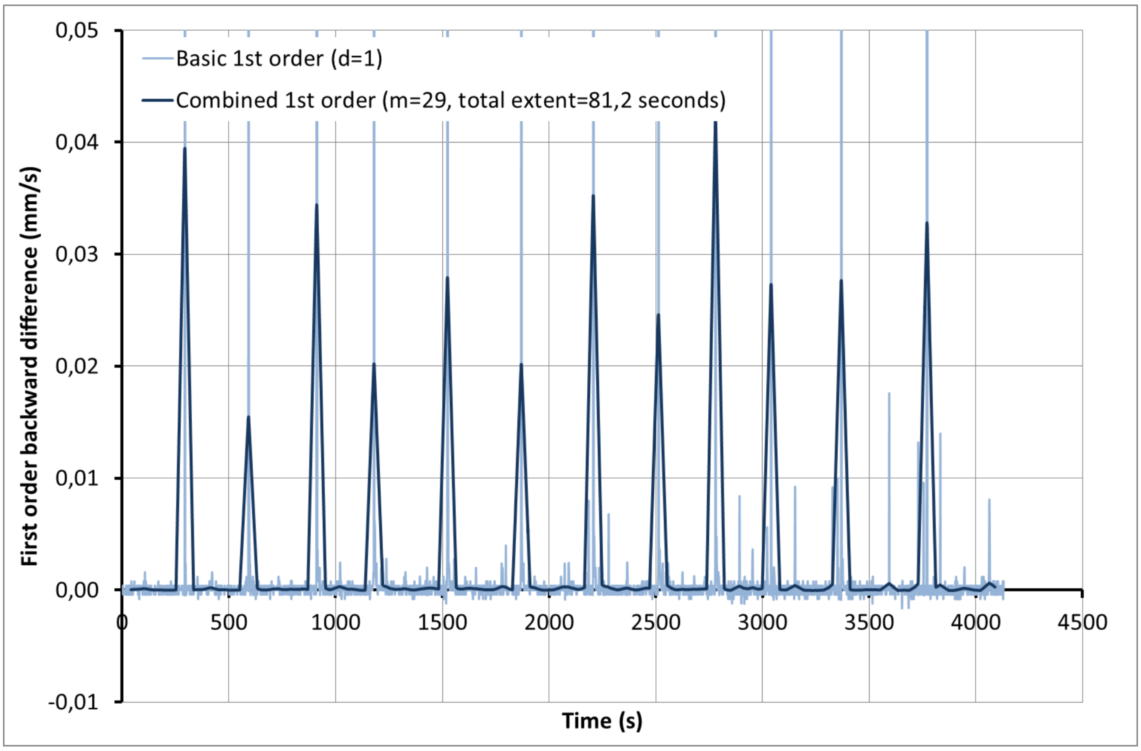



0,40
0,35
0,30
0,25
0,20
0,15
0,10
0,05
0,00
-0,05
First order backward difference (mm/s)
Basic 1st order (d=1)
Combined 1st order (m=11, total extent=116,6 seconds)
0
500
1000
1500
2000
2500
3000
3500
4000
Time (s)

Example 2

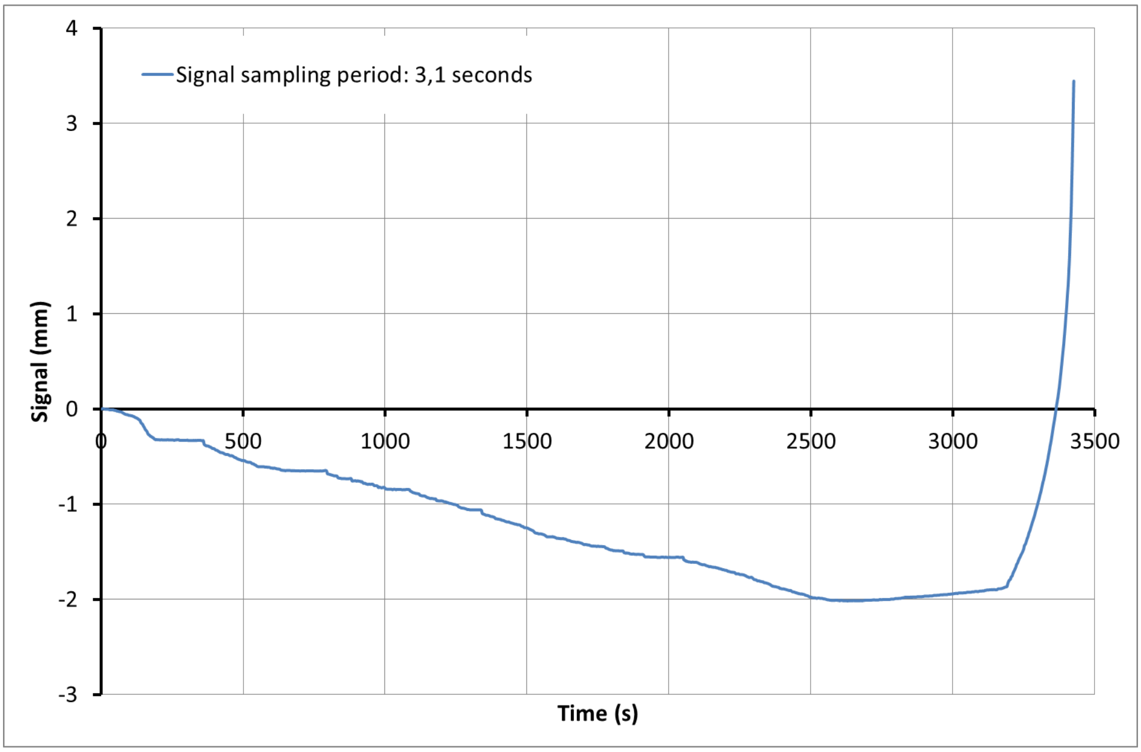

Signal sampling period: 1,4 seconds
Signal (mm)
Time (s)
16
14
12
10
8
6
4
2
0
-2
0
500
1000
1500
2000
2500
3000
3500
4000
4500

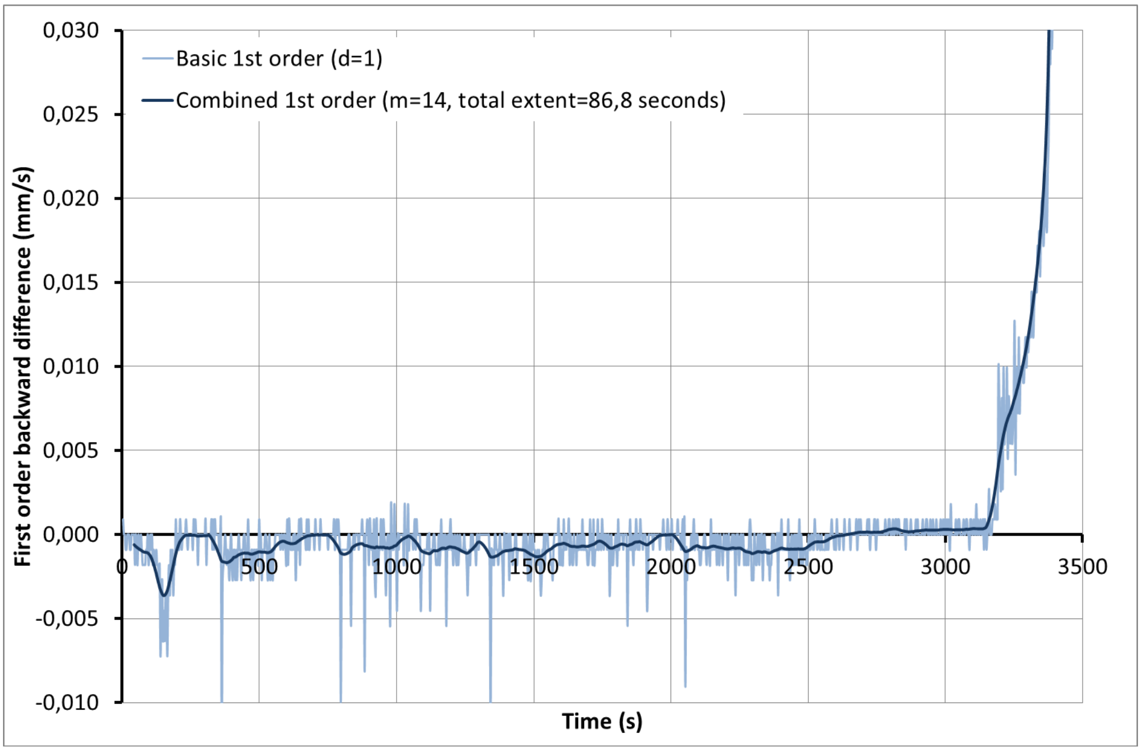

Basic 1st order (d=1)
Combined 1st order (m=9, total extent=25,2 seconds)
First order backward difference (mm/s)
Time (s)
0,05
0,04
0,03
0,02
0,01
0,00
-0,01
0
500
1000
1500
2000
2500
3000
3500
4000
4500



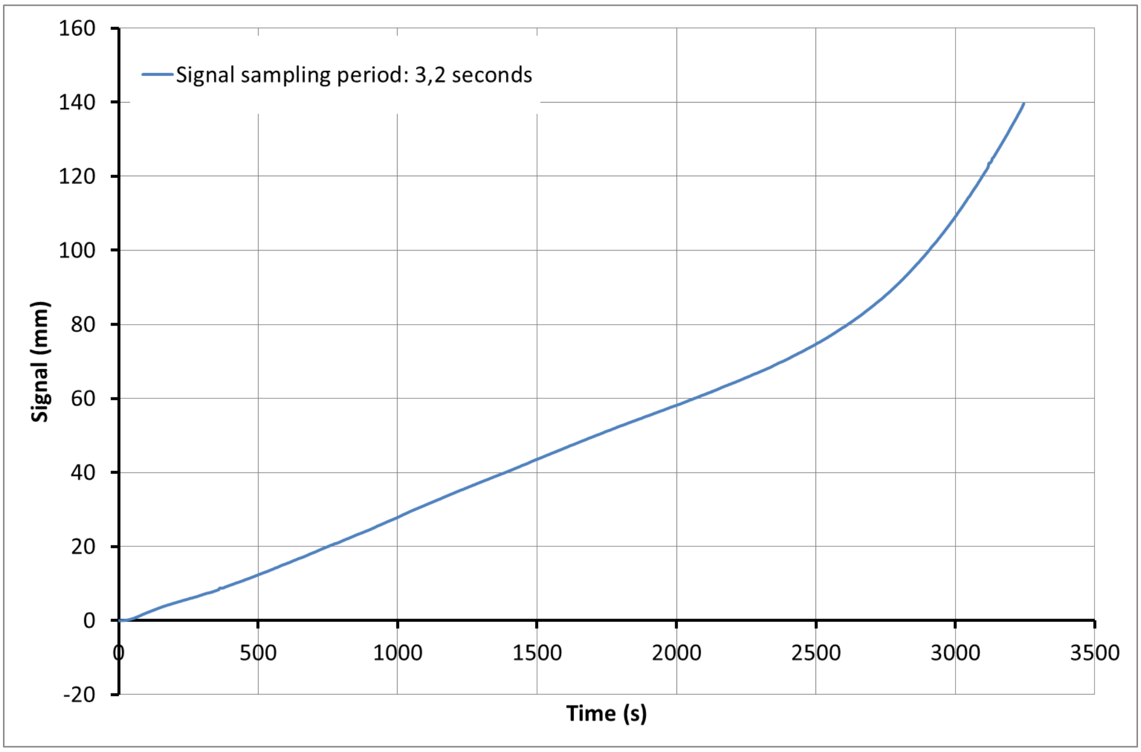
This example shows how difficult is the differentiation of signals with sharp jumps, and even discontinuous. In such cases, the finite difference methods reveal rather inefficient. The most suitable approach seems then to be the “total-variation regularization” (Chartrand, Numerical Differentiation of Noisy, Nonsmooth Data, 2011).

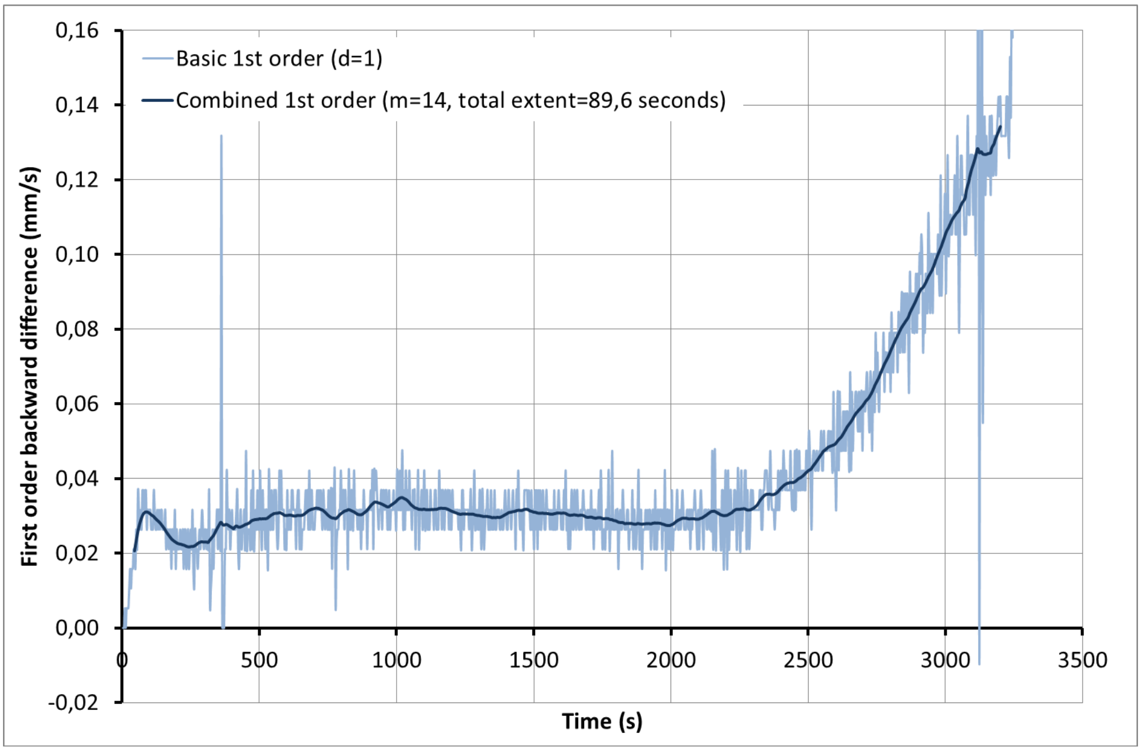
<u>EXAMPLE 3</u>


Signal sampling period: 3,1 seconds
Signal (mm)
Time (s)
4
3
2
1
0
-1
-2
-3
0
500
1000
1500
2000
2500
3000
3500

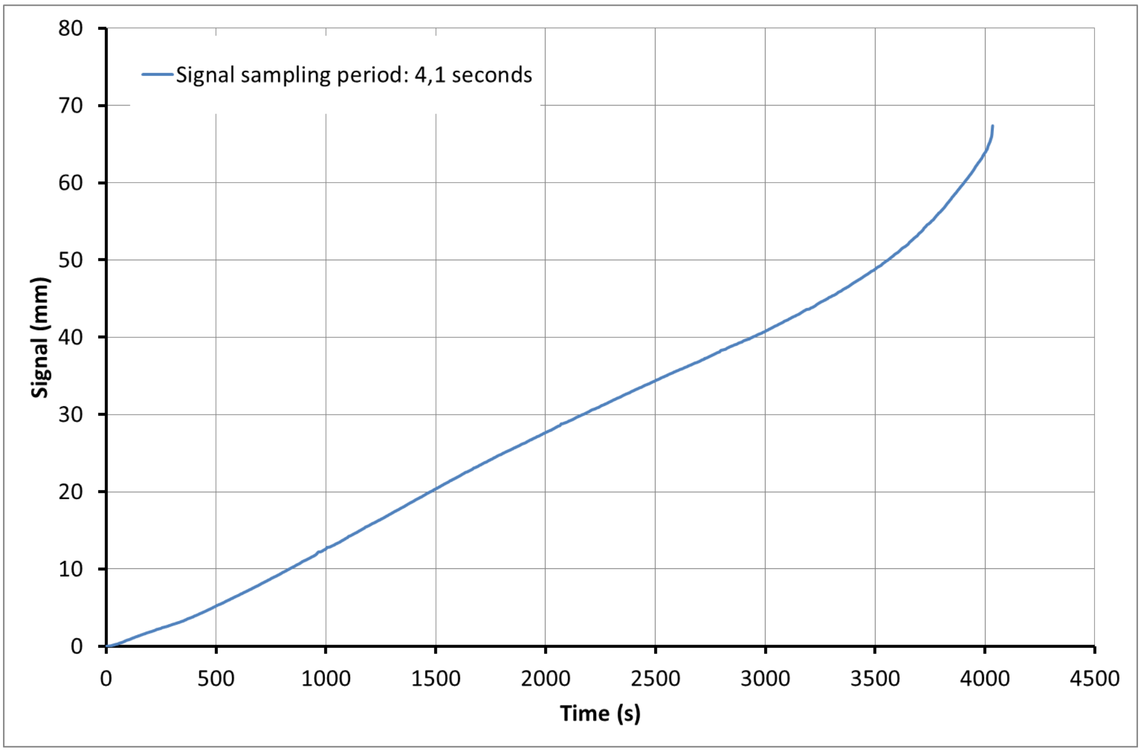

Basic 1st order (d=1)
Combined 1st order (m=14, total extent=86,8 seconds)
First order backward difference (mm/s)
Time (s)
0,030
0,025
0,020
0,015
0,010
0,005
0,000
-0,005
-0,010
0
500
1000
1500
2000
2500
3000
3500

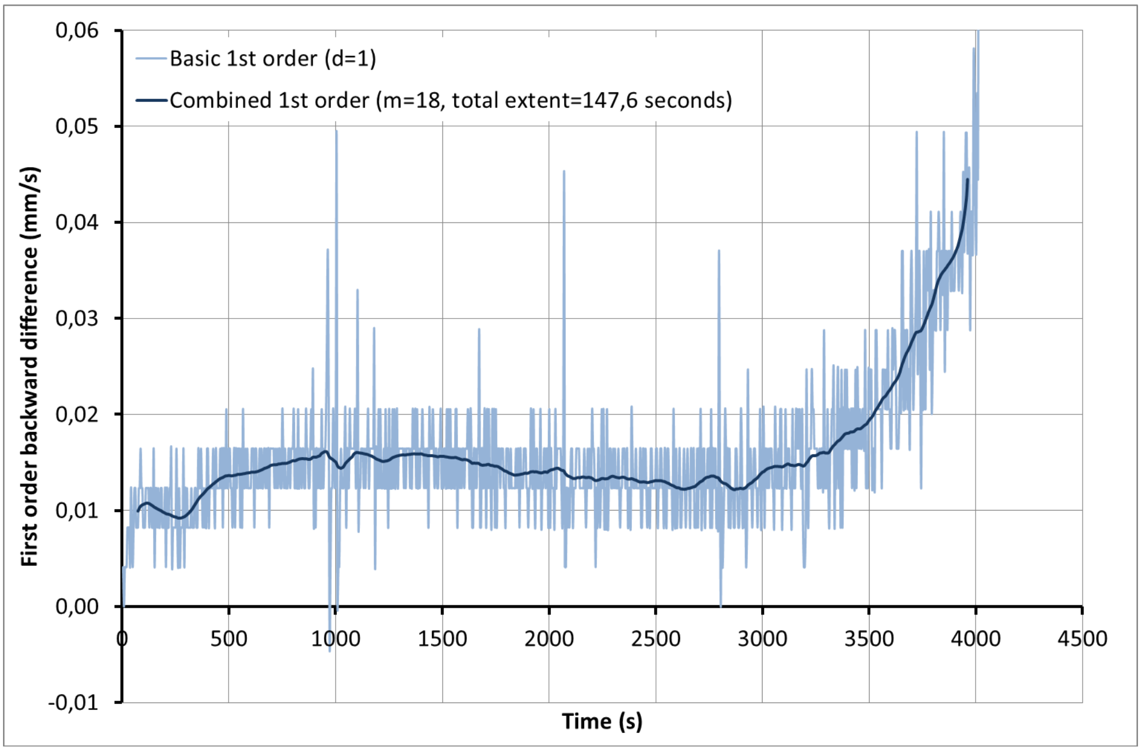
<u>EXAMPLE 4</u>


Signal sampling period: 3,2 seconds
Signal (mm)
Time (s)
160
140
120
100
80
60
40
20
0
-20
0
500
1000
1500
2000
2500
3000
3500

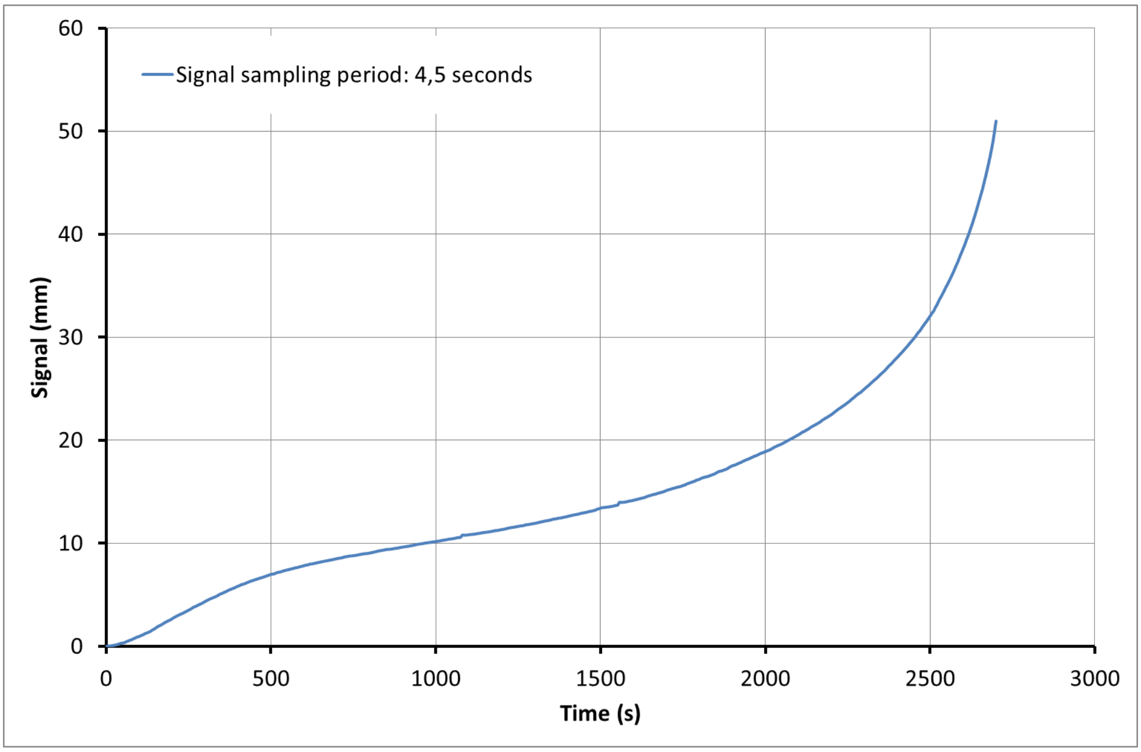

Basic 1st order (d=1)
Combined 1st order (m=14, total extent=89,6 seconds)
First order backward difference (mm/s)
Time (s)
0,16
0,14
0,12
0,10
0,08
0,06
0,04
0,02
0,00
-0,02
0
500
1000
1500
2000
2500
3000
3500

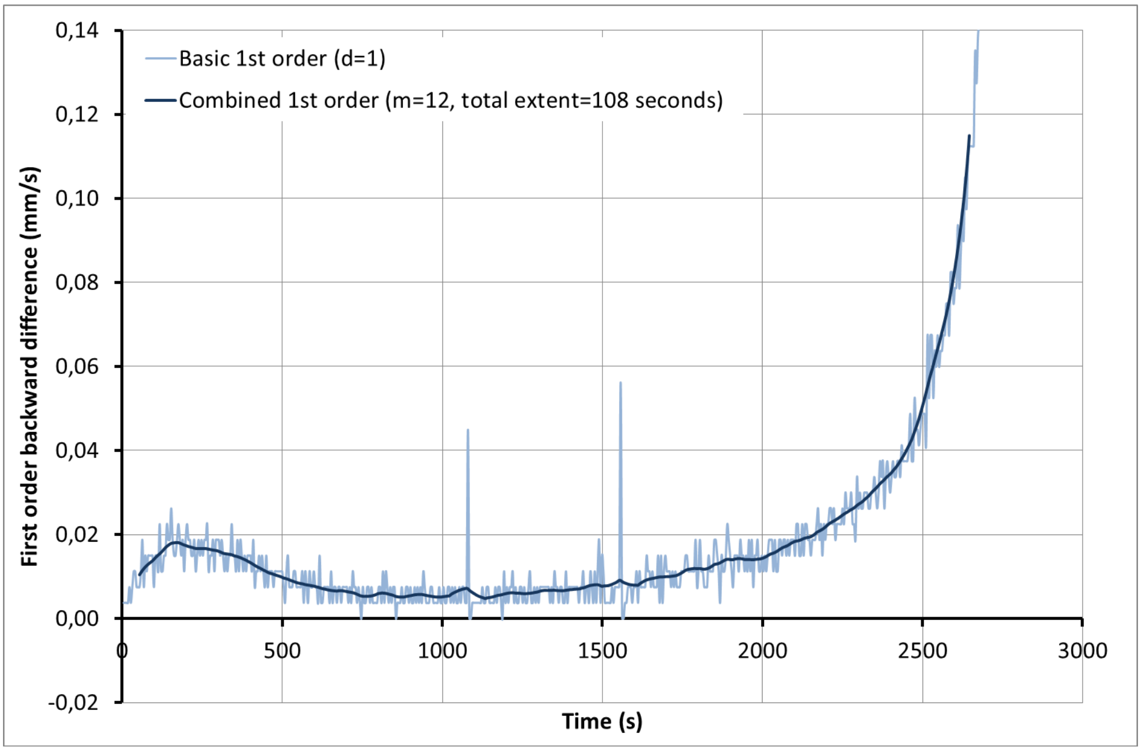
EXAMPLE 5


Signal sampling period: 4,1 seconds
Signal (mm)
Time (s)
0
10
20
30
40
50
60
70
80
0
500
1000
1500
2000
2500
3000
3500
4000
4500

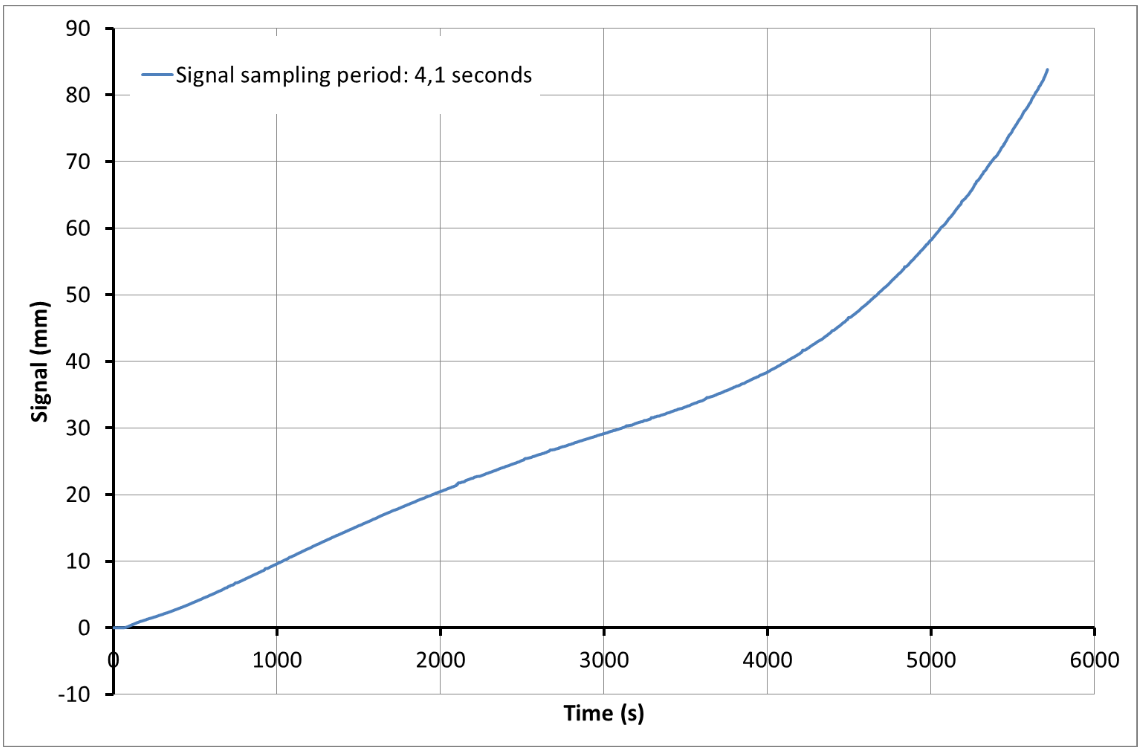

Basic 1st order (d=1)
Combined 1st order (m=18, total extent=147,6 seconds)
First order backward difference (mm/s)
Time (s)
-0,01
0,00
0,01
0,02
0,03
0,04
0,05
0,06
0
500
1000
1500
2000
2500
3000
3500
4000
4500

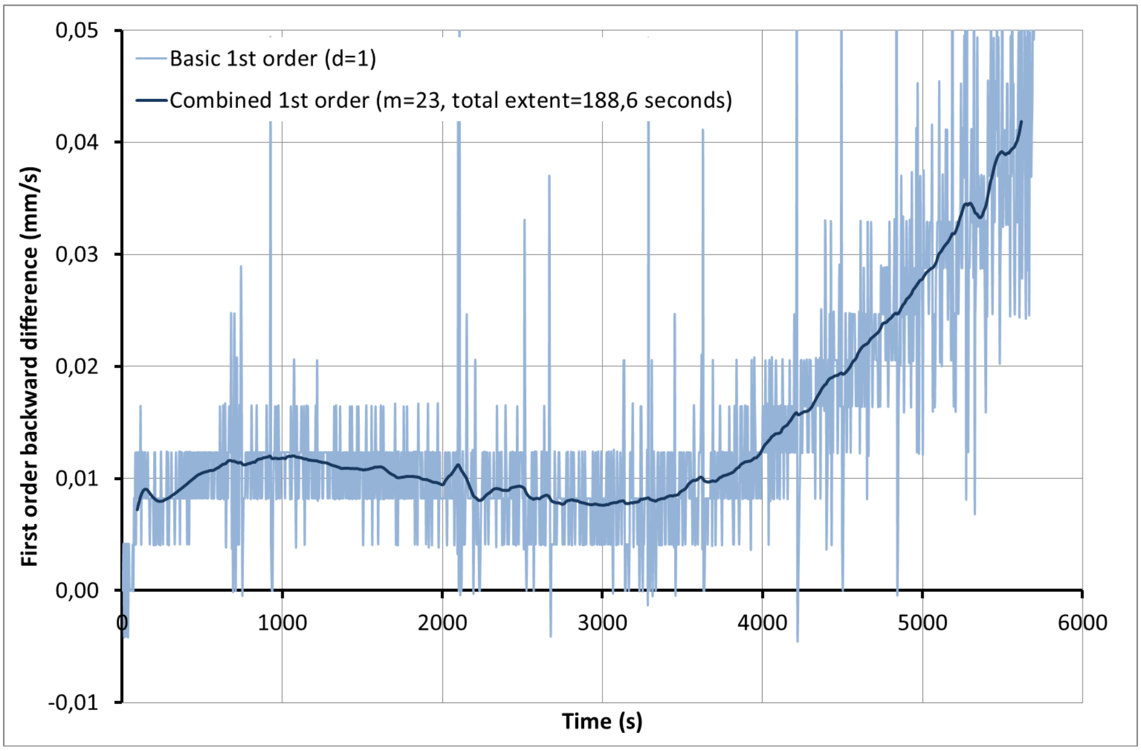
EXAMPLE 6


Signal sampling period: 4,5 seconds
Signal (mm)
Time (s)
0
10
20
30
40
50
60
0
500
1000
1500
2000
2500
3000

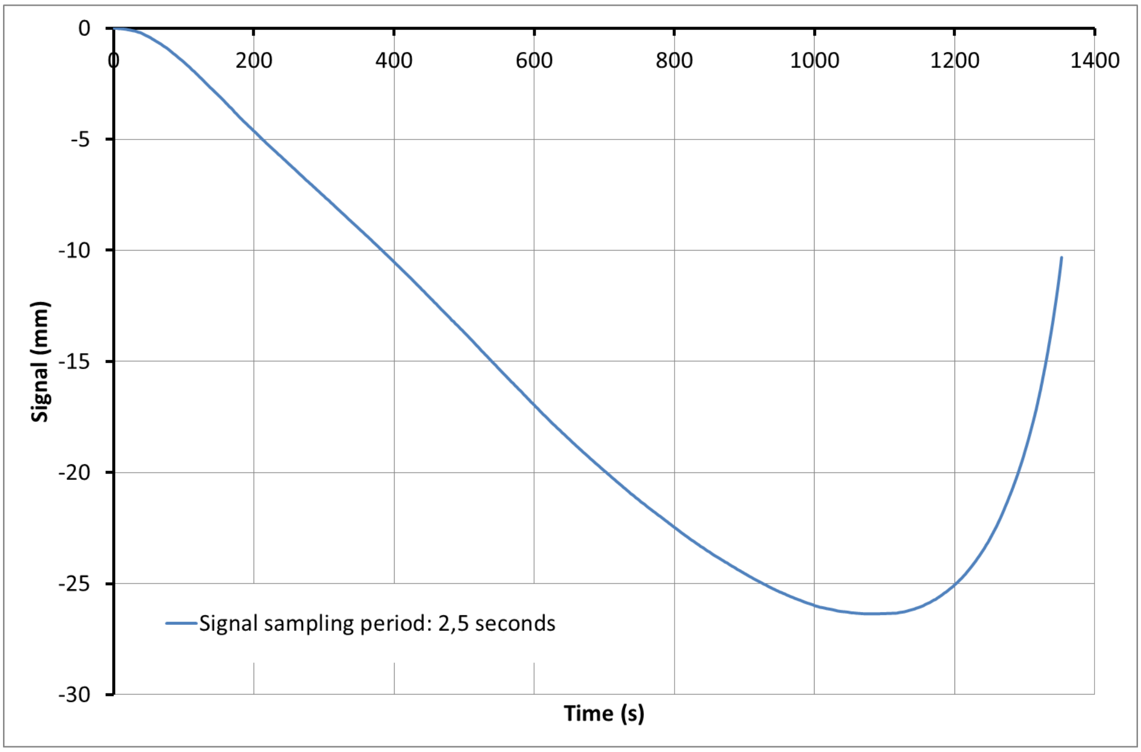

Basic 1st order (d=1)
Combined 1st order (m=12, total extent=108 seconds)
First order backward difference (mm/s)
Time (s)
-0,02
0,00
0,02
0,04
0,06
0,08
0,10
0,12
0,14
0
500
1000
1500
2000
2500
3000

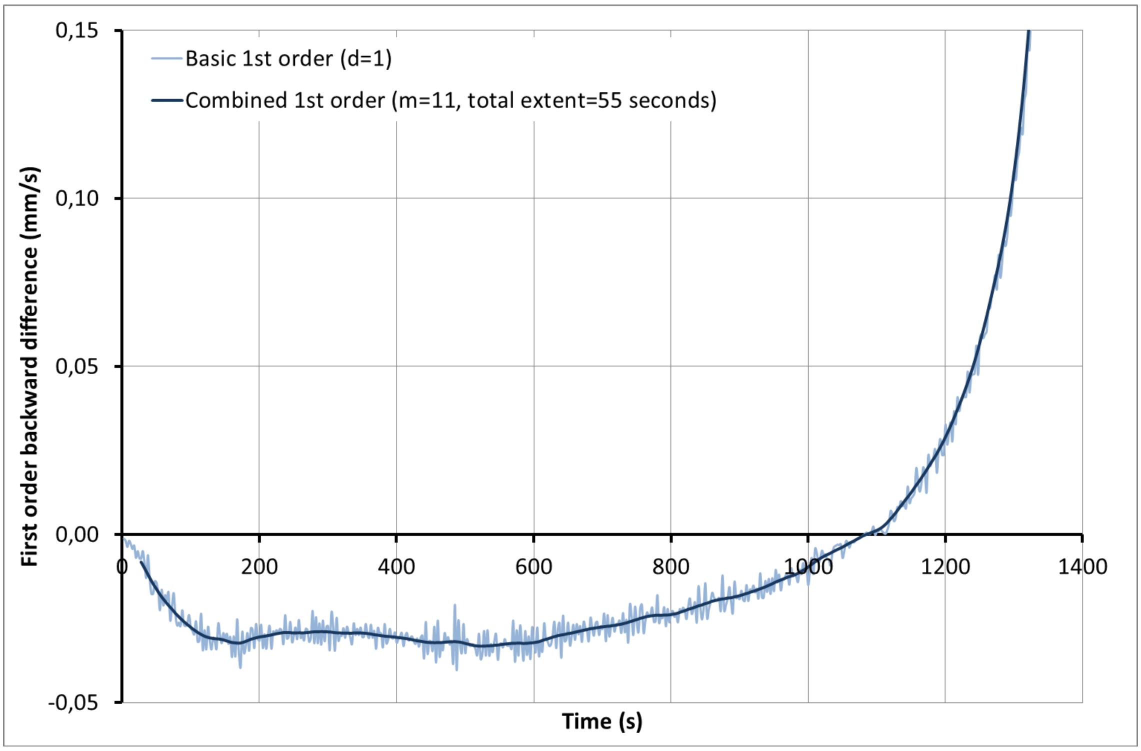
EXAMPLE 7


Signal sampling period: 4,1 seconds
Signal (mm)
Time (s)
90
80
70
60
50
40
30
20
10
0
-10
0
1000
2000
3000
4000
5000
6000

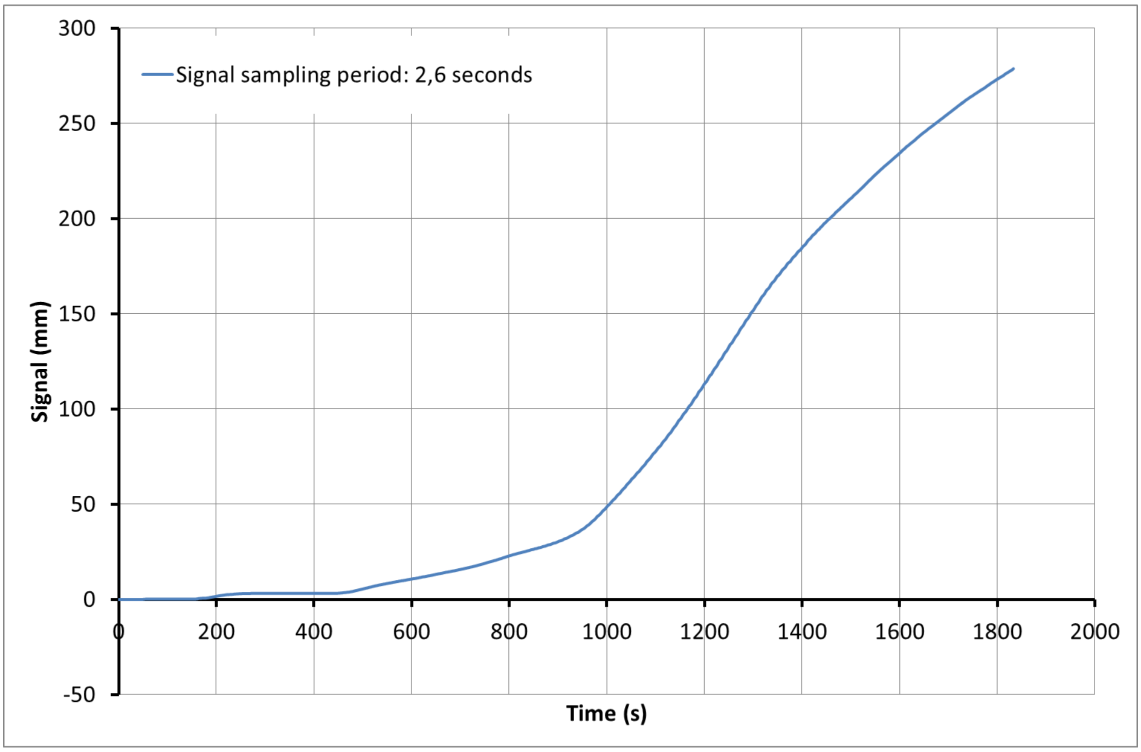

Basic 1st order (d=1)
Combined 1st order (m=23, total extent=188,6 seconds)
First order backward difference (mm/s)
Time (s)
0,05
0,04
0,03
0,02
0,01
0,00
-0,01
0
1000
2000
3000
4000
5000
6000

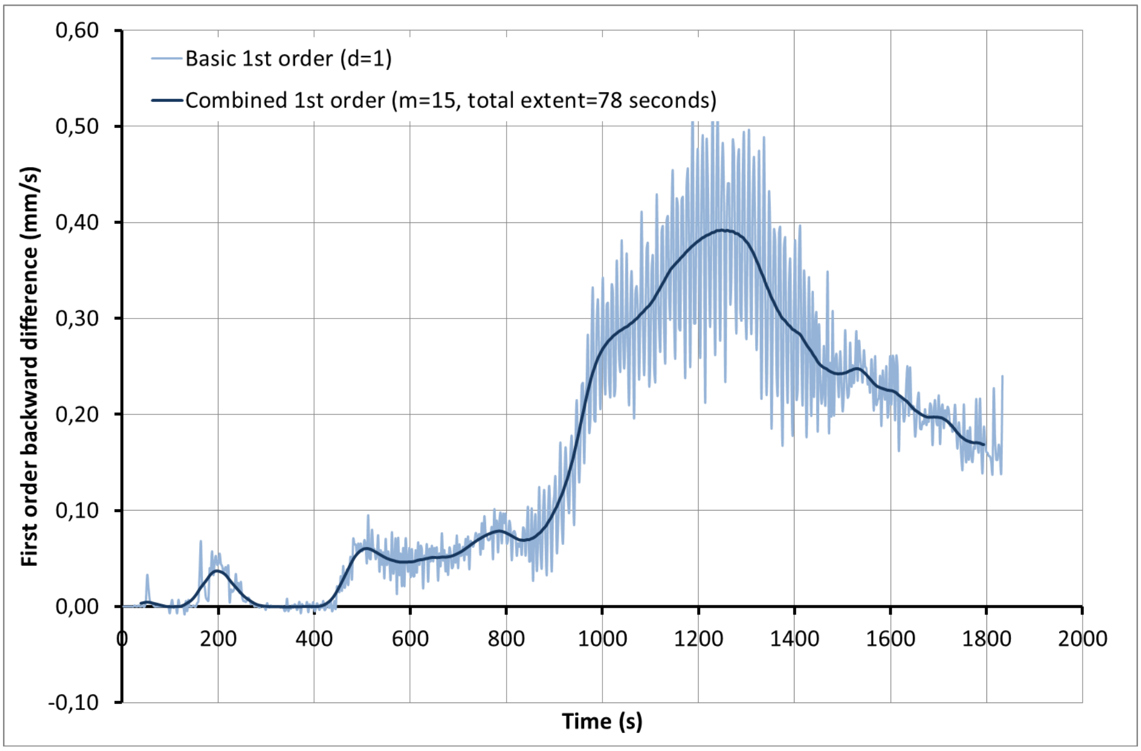
EXAMPLE 8


Signal (mm)
0
-5
-10
-15
-20
-25
-30
0
200
400
600
800
1000
1200
1400
Signal sampling period: 2,5 seconds
Time (s)


First order backward difference (mm/s)
0,15
0,10
0,05
0,00
-0,05
0
200
400
600
800
1000
1200
1400
Basic 1st order (d=1)
Combined 1st order (m=11, total extent=55 seconds)
Time (s)

EXAMPLE 9


Signal sampling period: 2,6 seconds
Signal (mm)
Time (s)
300
250
200
150
100
50
0
-50
0
200
400
600
800
1000
1200
1400
1600
1800
2000


Basic 1st order (d=1)
Combined 1st order (m=15, total extent=78 seconds)
First order backward difference (mm/s)
Time (s)
0,60
0,50
0,40
0,30
0,20
0,10
0,00
-0,10
0
200
400
600
800
1000
1200
1400
1600
1800
2000