Economic statistical design of nonparametric control charts

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Chapter 1

Introduction

It is often critical to control and improve the quality of the products and processes of a company. Indeed, the choice of a consumer for one product or another is more and more influenced by the quality of the competing products (Montgomery, 2007). Quality control and quality improvement have thus become major concerns for the companies, as they constitute key factors to success. This control and improve necessity is important for most companies, be they manufacturing, distributing or transportation companies, as well as healthcare or financial services providers.

There exist several ways to define, and consequently to control and evaluate, the quality. A traditional definition of quality considers quality as a measure of '*fitness for use*' (Montgomery, 2007). This definition implies two facets of the quality: quality of design and quality of conformance. However, throughout the years, the first aspect slowly faded away, and the second became more important, leading to an approach of the quality where the compliance with the specifications was prominent. A more modern definition states that '*quality is inversely proportional to variability*' (Montgomery, 2007, p. 6). This definition is motivated by the fact that the processes run by a company should ideally be stable and repeatable, and should operate at some optimal predefined level. Indeed, any deviation from the optimal level, or any variability, will usually induce additional costs. The companies thus desire to maintain their processes under control, and this implies that the process operates under little variability. This definition allows to express quality in monetary terms, which are more easily understandable by everyone in the firm. In accordance with the modern definition of quality, Montgomery (2007, p. 7) defined quality improvement (QI) as being '*the reduction of variability in processes and products*'.

Statistical process control (SPC), or statistical quality control, is a collection of problem-solving tools that uses statistical methods, among others, to reduce the variability required to achieve process stability. Other tools, such as design of experiments and acceptance sampling, are also useful to control and improve the quality, but will not be discussed here. SPC, design of experiments and acceptance sampling, constitute only the technical basis of quality improvement, and any QI initiative must be implemented as part of a larger improvement program in order to maximize its efficiency. Indeed, the success of quality improvement does not depend only on the correct use of SPC techniques. It must become part of the culture of the company, and has to drive the management system. It is really crucial for the success of these techniques that the management is aware of its responsibility in the QI implementation as a full and continuous task. Several improvement frameworks considering the quality improvement task as a whole have been developed throughout the years by several researchers and companies. As examples, we can cite the plan-do-check-act (PDCA) cycle promoted by W. Edwards Deming, the total quality management (TQM) framework, or the six-sigma framework developed by Motorola. In this work, we will only be interested in the technical aspects of SPC, and will not discuss larger QI frameworks.

In statistical process control, there exist seven tools that are of particular interest, and that are often called the ‘magnificent seven’: the histogram, the check sheet, the Pareto chart, the cause-and-effect diagram, the defect concentration diagram, the scatter diagram and the control chart. The first six tools are somewhat basic, though very powerful when all the tools are used simultaneously. In this work, we will be interested in the more sophisticated control chart, which is both more complex to understand and to set up, but which relies on sound mathematical principles. The strong mathematical foundation of the control charts, as well as their simplicity of implementation and of use, probably explains the success of this tool as a key component of SPC.

Control charts offer a way to control whether a process operates under normal predefined conditions or not. In the first case, the process is said to be ‘in-control’ (IC), while it is considered to be ‘out-of-control’ (OC) in the latter. Control charts determine the state of a process from the observation of a variable, called quality characteristic, whose value is observed at some time steps. This variable is assumed to be a random variable whose distribution is unknown in most cases. Traditionally, the control chart determines that the process is out-of-control, i.e. gives an out-of-control signal, when the output of a function computed from a bunch of observations of the quality characteristic does not fall between acceptable bounds.

Control charts can be divided into two categories: the parametric and the nonparametric charts. The parametric control charts make strong probabilistic assumptions on the variable to control. More specifically, these charts generally assume that the variable under control follows a predefined distribution. The nonparametric control charts, on the other hand, make weaker probabilistic assumptions. For example, they may assume that the distribution is symmetric. Parametric control charts usually offer a better control of the process when the distribution of the variable is the one that was assumed by the chart. However, when the variable follows a different distribution, the performances of the chart might deteriorate dramatically. Because of their weaker assumptions, nonparametric charts are more robust than their parametric counterparts to changes in the distribution, but this robustness is achieved at the expense of some loss in the performances when the distribution is perfectly known. The efficiency of a control chart is thus conditioned by how well the real distribution of the variable under control fits the probabilistic assumptions of the chart. Choosing a control chart that fits the distribution of the variable is thus of utmost importance in order to efficiently control the process. The difficulty in choosing the type of control chart that is going to be used for an application follows from the lack of information that most of the time affects any process to control.

Besides their type (parametric or nonparametric), there exist a few parameters that are common to all control charts, and that must be determined in order to practically implement them. In the beginning, the parameters were chosen based on some heuristics that were known to perform acceptably well in practice. Later on, more sophisticated methods were designed to choose the good parameter values. These approaches are referred to as statistical design and economic design. These methods find values of the parameters in such a way that some statistical or cost guarantees are respectively ensured. Economic statistical design (ESD) is a method that combines both ideas in a single approach. ESD allows to find good design parameters for a given control chart by minimizing the expected cost of operating the process, while imposing statistical constraints on the ability of the chart to detect quality characteristic deviations from its optimal value.

When a control chart for a given application has to be implemented, the type of the control chart must first be chosen according to the available a priori knowledge. However, this knowledge might be scarce, incomplete, or wrong. In that case, choosing an inappropriate parametric control chart might result in very poor control performances. For this reason, when the available a priori knowledge is deemed unreliable, it is sometimes preferable to use nonparametric charts instead of parametric ones.

This work focuses on the study of the economic statistical design of nonparametric control charts with an application to a standard delivery chain process. To our knowledge, this is the first time that ESD is applied to nonparametric charts.

The remainder of this document is organized as follows. First, we give, in Chapter 2 and Chapter 3, a brief introduction to statistical process control and to control charts, in order to understand their basic principles. Chapter 4 is focused on the review of the most famous control chart design techniques. Then, Chapter 5 explains the economic statistical design of nonparametric control charts, which is the core of this work. Chapter 6 next describes in detail the experiments that we have carried out to validate our method, and reports the obtained results, together with their analysis. Finally, Chapter 7 draws the general conclusions of this report and proposes some lines of future work.

Chapter 2

Introduction to statistical process control

The processes run by a company should generally operate under little variability. Manufactured parts of an engine are all expected to have the same dimensions. The passenger's experience aboard the plane of an airline is expected to be constant throughout the flights. It is desirable that the delivery chain of a company does not suffer from delays. If a part does not have the right dimensions, another one has to be used, or the faulty part must be corrected so that it fits into the rest of the engine. If a passenger does not have a meal because too few dishes were taken on board, the passenger will not be satisfied and the airline might have to financially compensate for its error. Similarly, delays in delivery chains may often lead to financial compensations.

These examples show why companies desire that their processes are stable and reliable in order to minimize their costs and to maximize the quality of the provided services. Unfortunately, no process can be perfectly stable, and no process can consistently provide the same outputs. Because of this inevitable variability, control methods being able to effectively detect irregular behaviors are required in order to take correcting actions as soon as possible. If the errors occur often, understanding their causes and why these errors appear is important in order to improve the processes to further reduce their variability.

Statistical process control (SPC) is a collection of tools that have been designed to control and improve the processes of a company. Some SPC tools are useful to control the processes in order to reduce their variability, while others are useful to understand how processes work and which causes can affect them in order to further improve the way processes operate.

The rest of this Chapter is organized as follows: we first present the causes for variability in the processes and then describe the basic tools of statistical process control.

2.1 Variability in the processes

As mentioned earlier, the desired output of a process should, in general, be stable. This means that the value of the process output is expected to always be the same. This value can be called the optimal or target value. For example, it is clear in the manufacturing industry that the produced parts are expected to all have the same dimensions, e.g. the diameter of a cylindrical part. However, it is foolish to imagine that any process can be indefinitely stable. The consequence of the variability is that the output of any process is actually a random variable that takes its values around the target value. This random variable follows some probability distribution that is, in most cases, unknown. Because the probability distribution of the output of a process is unknown, the simplest way to characterize the process variability is to use a mean and a standard deviation, where the mean is generally the desired target value.

In the context of economy, the process variability was first studied by Shewhart (1931). He theorized that there exist mainly two types of variations that can affect a given process. In the rest of the section, a brief description is given of these two types of perturbations that can disturb the output of a process.

The first type was referred to by Shewhart (1931) as 'chance causes of variation', although some authors prefer nowadays the term 'common causes' (Montgomery, 2007). Chance causes designate the natural variability that affects any process and that, no matter how well maintained and how carefully thought the process is, will always be present. Because these causes are essentially unavoidable and part of any process, we will say that a process is in the statistical control state, or 'in-control' (IC), when the only causes that affect it are chance causes. The possible chance causes and their importance determine the in-control mean and in-control standard deviation of the process, respectively denoted μ_{ic} and σ_{ic}. These two values characterize the in-control distribution, which is the normal and acceptable distribution of the output of the process when it operates under optimal conditions.

Shewhart (1931) also identified 'assignable causes of variation', which are sometimes called 'special causes' according to a more recent terminology (Montgomery, 2007). Those causes usually induce an intolerable level of variation that deteriorates process performances to an unacceptable level. The sources of assignable causes are many and depend on the process of interest. An assignable cause can affect the output of a process in several ways. It can for example completely change the probability distribution of the process, i.e. the mean, the standard deviation and the shape of the distribution. Or it can simply change either the mean or the standard deviation (or both), without changing the shape of the distribution of the random variable. When assignable causes impact the variability of a process, we say that the process is in an 'out-of-control' (OC) state, in which the distribution of the output is characterized by the out-of-control mean μ_{oc} and the out-of-control standard deviation σ_{oc}.

Figure 2.1 illustrates the probability distributions of the output of a process in several cases. The upper left graph illustrates an in-control distribution of the output, while the upper right compares an in-control and an out-of-control distributions where the IC and OC standard deviations are equal, but the means are different. In the lower graphs, the IC and OC standard deviations are different. The lower left graph shows the case where IC and OC means are equal, while the lower right compares the IC and OC distributions when both the means and the standard deviations are different.

There is an important difference between the two types of causes. Chance causes are part of any process, and will always affect the process. They induce a level of variability that is acceptable due to their inevitable nature. Assignable causes, unlike chance causes, are, to

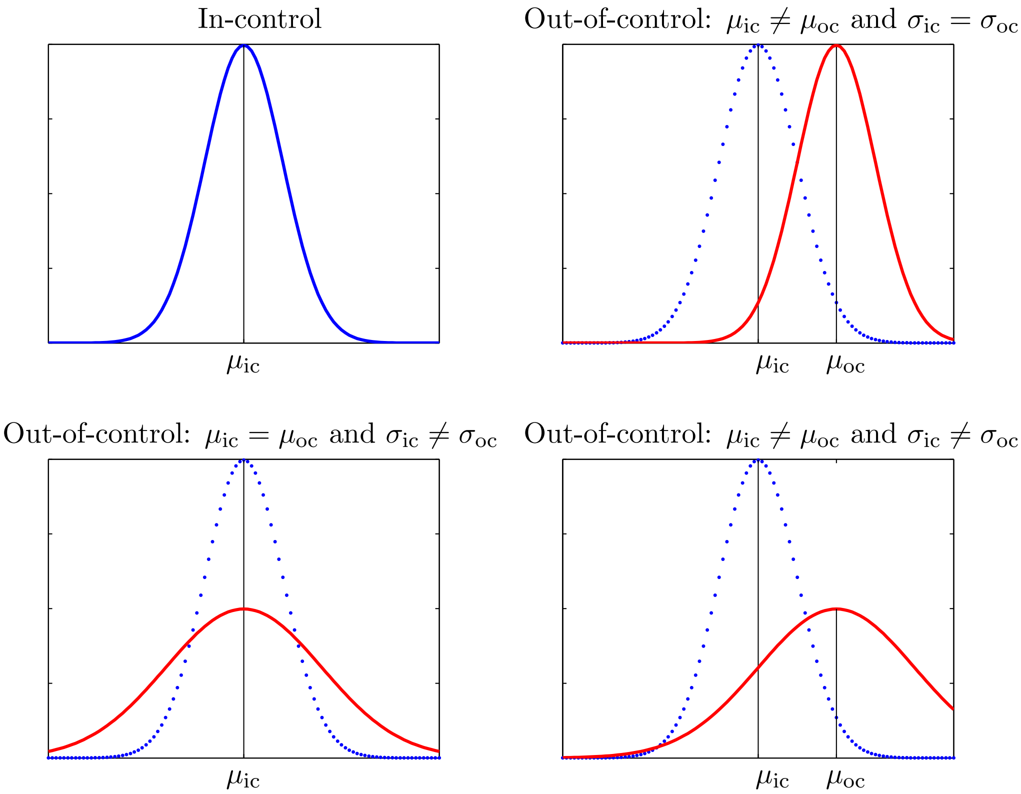


Figure 2.1: In-control and out-of-control probability distributions of the output of a process in several cases. The in-control distribution is drawn in blue on the graphs, while the out-of-control distributions are drawn in red.

some extent, avoidable and revertible. In order to maintain an efficient process that operates under optimal conditions, it is thus important to detect and correct assignable causes as soon as possible. Detecting that an assignable cause occurred is one of the major goals of statistical process control. Even if it is impossible to entirely eliminate the variability of a process, SPC offers useful tools to reduce it as much as possible.

There exist different types of variability that can affect a process in a given state, say the IC state. The first type of variability is called stationary. This type of variability implies that the distribution of the process does not vary with time. More specifically, this implies that the process output varies around some predefined mean. In the stationary case, the successive outputs can still be either uncorrelated or autocorrelated. Uncorrelated here means that one output is independent from the previous one, while autocorrelated indicates that successive observations are dependent. The first case is easier to analyze thanks to the independence assumption. The other type of variability that can affect a process is called nonstationary variability, where the distribution of the process can change over time. This type of variability is usually more complicated to deal with. In this work, we will consider the simplest case of stationary uncorrelated data.

2.2 The magnificent seven

The so-called 'magnificent seven' are the main tools of SPC (Montgomery, 2007). The magnificent seven are: the histogram, the check sheet, the Pareto chart, the cause-and-effect diagram, the defect concentration diagram, the scatter diagram and the control chart. The seven tools together constitute a homogeneous framework for variability reduction and quality improvement.

The first six tools are quite simple to understand and to use, and we will briefly describe them here. The control chart is, on the other hand, a bit more subtle and, as it is more sophisticated, it is more interesting to study and necessitates more time to get used to. The description of the control charts is left to the next chapter.

While control charts are mainly used to control a given process, they also provide useful information for the understanding of the process and consequently for its improvement. On the other hand, the other six tools tend to be more useful when it comes to the analysis and understanding of the process in order to identify the most frequent or crucial errors.

Note that there exists another set of tools, called the 'seven basic tools of quality' (Imai, 1986), that slightly differ from the magnificent seven. The seven basic tools do not contain the defect concentration diagram, but contain instead the flow chart. In this report, we only describe the magnificent seven.

2.2.1 The histogram

A histogram (PMI, 2004) is a bar chart where each bar represents a variable and where the height of the bar models the frequency of that variable. The histogram is an approximation of the real distribution of the variables. This graph can be used to identify the most frequent errors. Figure 2.2 illustrates a histogram representing the frequency of the number of arrivals per minute of a given process. This graph is just an illustration and does not represent a real process.

2.2.2 The check sheet

The check sheet (Montgomery, 2007) is a document that summarizes historical operational data. Basically, the check sheet takes the form of a table in which some operational data is reported. There exist different types of check sheets depending on what they are meant to be used for. For example, we can create a check sheet to quantify defects by their type, by their location or by their cause. What will be reported on the two axes depends on the desired objective of the check sheet. Figure 2.3 illustrates a check sheet in the context of motor assembly where the defects are grouped by type.

The check sheet offers a broad view of operational data that facilitates the understanding of the process, and, more specifically, of the errors affecting it. However, in order for the check sheet to be as useful as possible in analyzing the errors of the process, any information that is important for the analysis must be reported on it. Such information is the type of

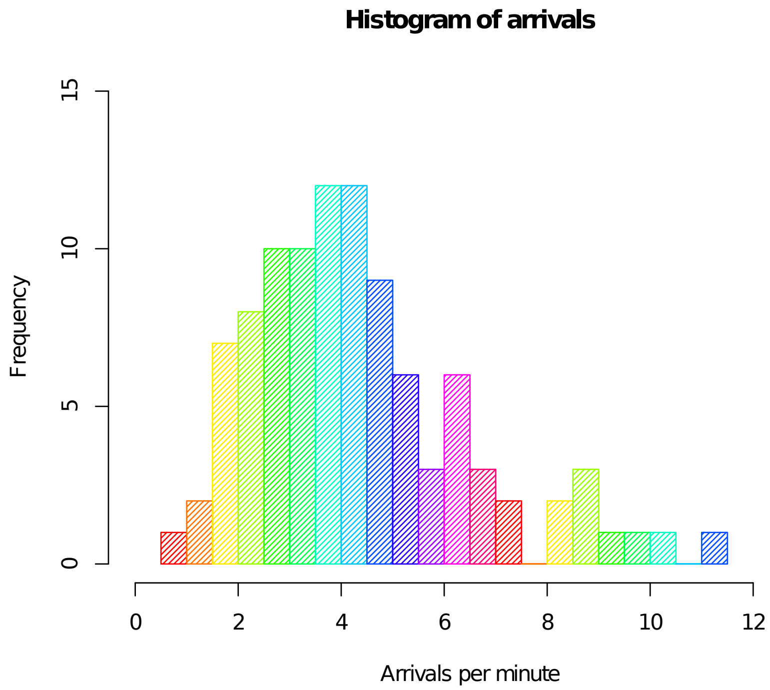


Figure 2.2: Example of a histogram (WikimediaCommons, 2010c).

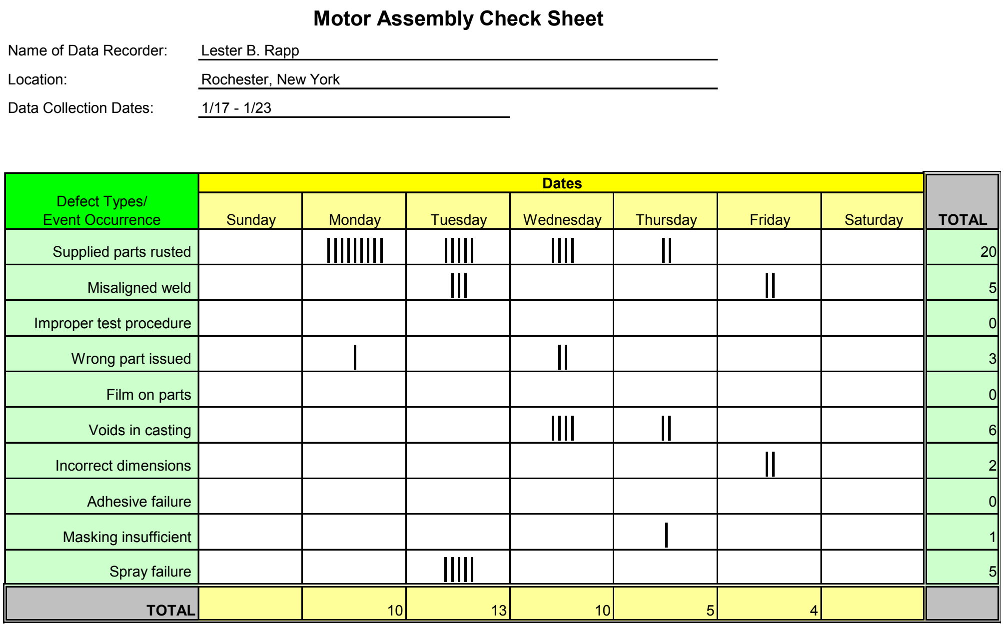
the collected data, the date, the analyst, the location, and any other information deemed important for the analysis.

2.2.3 The Pareto chart

A Pareto chart is a special histogram composed of bars and of a line, useful to easily visualize the most frequent elements in a given set. The elements are represented by bars whose heights are proportional to the frequencies of the elements associated with the bars. The bars are plotted on the graph by descending order in such a way that higher bars appear first. The line represents the cumulative frequencies from the most frequent element until the last one. Such a graph can be used to visualize the frequency of defects in a given process and to easily identify the most frequent ones. Note that the most frequent defect might not be the most important. Refinements of the graph are possible in order to take into account the relative importance of each defect. Figure 2.4 illustrates a Pareto chart in the context of the analysis of engine overheating causes.

2.2.4 The cause-and-effect diagram

When an error is detected, it is important to identify the potential causes that induced it. It is important both to correct the process in order to bring it back in-control, but also to improve the process in the long term. The cause-and-effect diagram (Montgomery, 2007) is a formal



Motor Assembly Check Sheet

Name of Data Recorder: Lester B. Rapp

Location: Rochester, New York

Data Collection Dates: 1/17 - 1/23

Defect Types/ Event Occurrence	Dates							TOTAL
	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	
Supplied parts rusted		IIIIIIIII	IIIII	IIII	II			20
Misaligned weld			III			II		5
Improper test procedure								0
Wrong part issued		I		II				3
Film on parts								0
Voids in casting				IIII	II			6
Incorrect dimensions						II		2
Adhesive failure								0
Masking insufficient					I			1
Spray failure			IIIII					5
TOTAL		10	13	10	5	4		

Figure 2.3: Example of a check sheet in the case of motor assembly (WikimediaCommons, 2010b).

tool that is used to identify the potential causes that produced a given error. The diagram is constructed by the team in charge of the quality improvement of the process, and requires a good knowledge of the process of interest. Figure 2.5 illustrates a generic cause-and-effect diagram.

2.2.5 The defect concentration diagram

The defect concentration diagram (Montgomery, 2007) consists in an image of the product showing different perspectives of the object. When a defect is identified somewhere on the object, it is reported on the defect concentration diagram at the location where it appears on the object. When this is done over a certain amount of units, patterns can be recognized and the location of the defects might help to identify their causes.

Figure 2.6 shows an example of defect concentration diagram on a fridge. The red parts indicate the locations of the defect. From this example, there appears to be a flaw in the process because all the defects are found at the same location. This will be helpful to understand the causes of the defects in order to repair them.

2.2.6 The scatter diagram

The scatter diagram (PMI, 2004) is a graphical tool that can reveal potential relationships between two variables, say x and y. The value of the two variables are collected in pairs during several runs of the process. The result is a set of pairs (x_i, y_i) where $i = 1 \dots n$ indicates which run of the process generated those values of the variables. The points (x_i, y_i) can then be

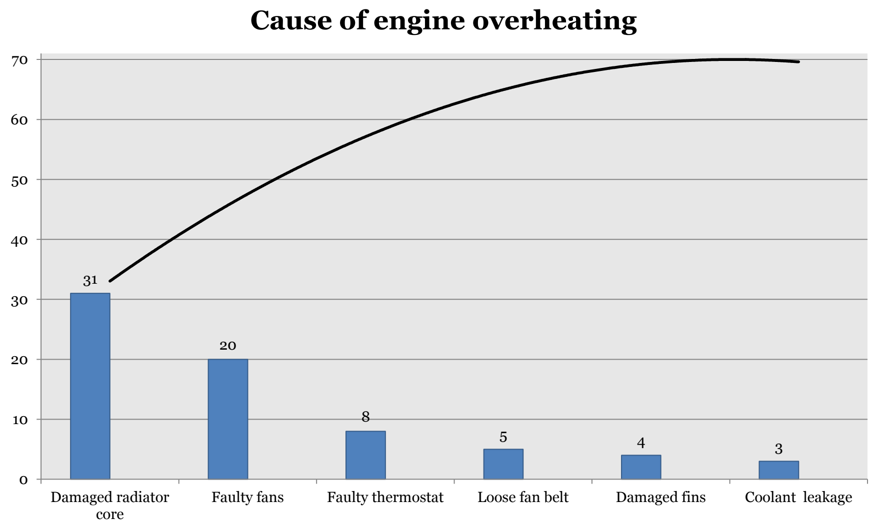


Figure 2.4: Example of a Pareto chart in the case of engine overheating (WikimediaCommons, 2013).

plotted on a graph. The shape of the plotted points indicate whether there is a possible relationship between the variables or not. However, one must be careful when analyzing the scatter plots, because correlation does not automatically imply causality. Several other variables might be involved in the causal relationship and the correlation observed between the studied variables might be a result of a causal relationship between other variables.

Figure 2.7 illustrates a scatter diagram between the process input and a given characteristic. In this case, we observe a negative correlation between the input and the characteristic. However, as we mentioned earlier, this does not necessarily imply that an increase of the input will decrease the value of the characteristic.

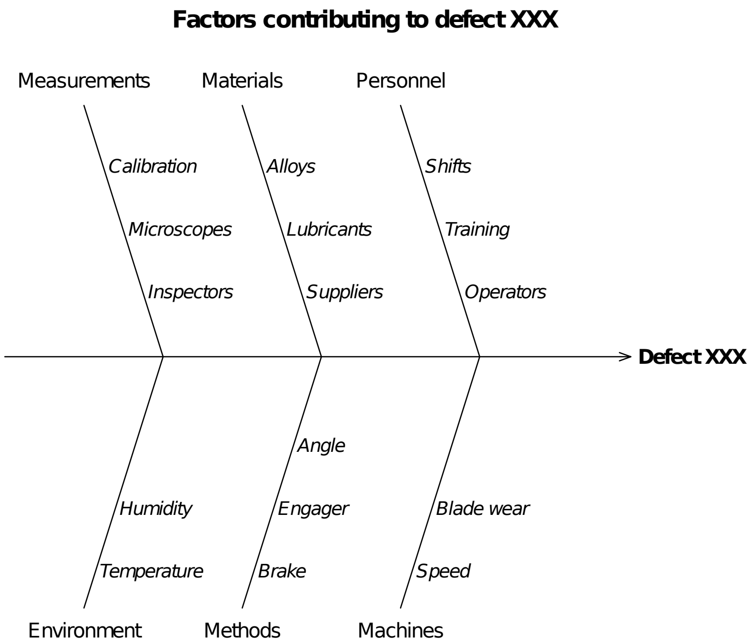


Figure 2.5: Example of a generic cause-and-effect diagram (WikimediaCommons, 2010a).

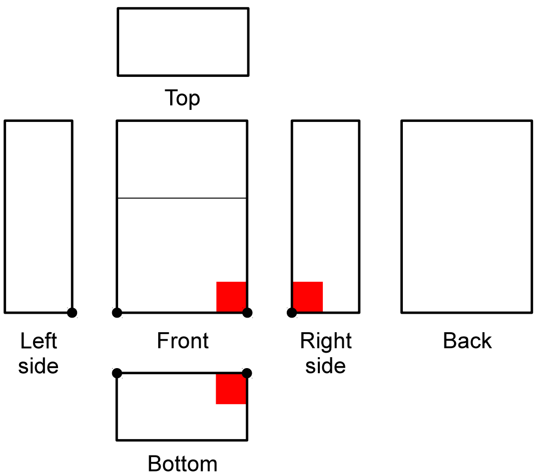


Figure 2.6: Example of a defect concentration diagram on a fridge.

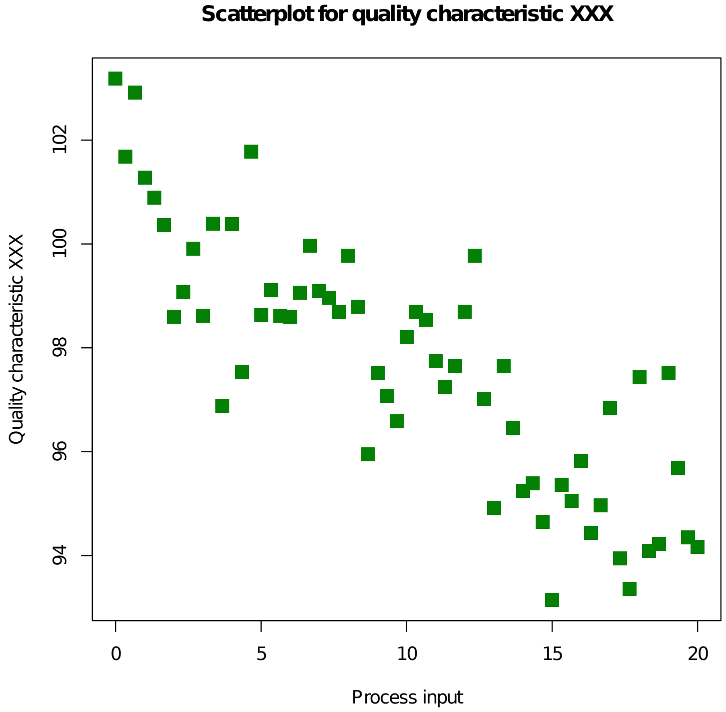


Figure 2.7: Example of a generic scatter diagram (WikimediaCommons, 2010d).

Chapter 3

Control charts

Control charts are statistical tools, first developed by Walter A. Shewhart, used to characterize whether a process is in-control (Montgomery, 2007). They can be used in many different ways, although they are mainly applied to monitor on-line processes. This chapter focuses on their definition, their basic mechanisms, and the description of parametric and nonparametric control charts.

3.1 Preliminaries

Control charts have been in use for many decades in many industries all around the world. The overall success of control charts can probably be best explained by their impact on productivity improvement that follows from other reasons, some of which have been identified by Montgomery (2007).

1. Control charts are efficient at defect prevention. The control charts help maintain the process in-control as much as possible.

2. Control charts are efficient at avoiding unjustified process corrections. The control charts are very effective in separating the assignable causes from the chance causes that affect any process.

3. Control charts give useful diagnostic information. The data reported on the chart generally conveys information about the causes that can hinder the performances of a process. This information can also give clues about how the process could be improved.

4. Control charts inform about process capability. The control charts provide information about many process characteristics that condition the capability of the process. This knowledge is very important in order to improve the process, and to better design the products as well as the processes themselves.

5. Control charts can be easily implemented with modern computers. Control charts are quite simple tools that, once understood, do not require a lot of efforts to be used. Consequently, the computations required to run a control chart can, in most applications, be performed in real time and on site on an ordinary personal computer.

3.1.1 Definition, basic mechanisms and assumptions

Montgomery (2007, p. 182) defines a control chart as a '*graphical display of a quality characteristic that has been measured or computed from a sample versus the sample number or time*'.

This definition illustrates the basic principles of control charts. The goal of a control chart is to control the value of some quality characteristic, or feature, of a product or process over time. In order to do so, the control chart requires that one collects samples of the output of the process at given moments. From each sample, a 'statistic', which is a value representing the characteristic that we want to control, is computed and then plotted on the control chart versus the corresponding sample number or versus time. The evolution of the statistic over time permits to check whether the process operates under acceptable conditions or not.

In the simplest case, there are three lines represented on a control chart. The 'central line' (CL) represents the desired value of the quality characteristic when there is no variability. In an ideal world, where the process is not affected by any type of variability, every computed statistic should fall on this line. As the world is imperfect, the 'upper control limit' (UCL) and the 'lower control limit' (LCL) represent the in-control zone in which the process is assumed to operate under acceptable conditions. If one point falls outside these limits, this might be an indication that the process is out-of-control. When the process is determined to be out-of-control, the control chart raises a signal, or an alarm. Figure 3.1 illustrates a traditional control chart with a central line and upper and lower control limits.

One important assumption with traditional control charts is that the data collected by the chart at successive samples is assumed to be stationary and uncorrelated. Some control charts are able to deal with autocorrelated data but will not be discussed here.

The samples taken at regular time intervals are central to control charts, and, as such, deserve some thinking. There exist typically two different sampling schemes. The first one is called 'snapshot sampling', and consists in constructing each sample by taking outputs of the process that occurred at the same time, or as temporally close as possible. This sampling scheme gives, in a way, an instant picture of the process, hence its name. The problem with such an approach is that, if the process goes out-of-control and then returns to an in-control state between two samples, then the assignable cause will most likely not be detected. The second sampling scheme, called 'random sample', is meant to detect such causes and consists in taking samples from the process over the entire sampling interval. However, random sampling generally increases the absolute values of the control limits, and detecting assignable causes might become more difficult.

It is clear that one must be vigilant when taking samples. The way the process is sampled can totally prevent the chart from detecting out-of-control behaviors, if the sampling scheme has not been thought carefully and if the characteristics of the process are not factored in. The concept of rational subgrouping (Montgomery, 2007) defines how a good sampling scheme should behave. According to rational subgrouping, the samples should be taken in such a way that the probability of assignable causes occurring between samples is maximized, while the probability of the same causes occurring within samples is minimized. Following this principle to the letter would imply that snapshot sampling is the best sampling scheme, and

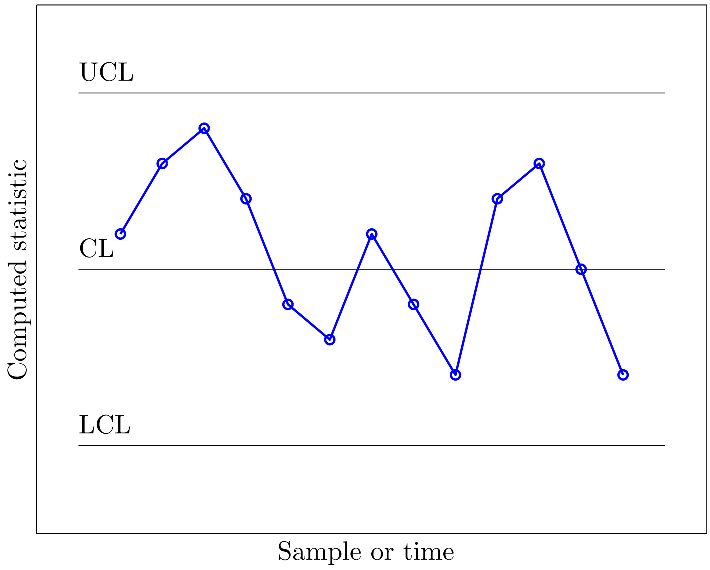


Figure 3.1: Example of a traditional control chart.

that changing the sampling interval is the best way to deal with assignable causes that can happen and then disappear between two consecutive samples.

3.1.2 A more detailed description of control charts mechanisms

In this section, we give a more formal and detailed description of how control charts actually work.

Assume an operator is controlling some quality characteristic x, i.e. some measure of the output of a process $\mathcal{P}$, which is represented by a real number. Let $\mathcal{C}$ be a control chart used by the operator to control if the values x remain in an acceptable range during the course of the operations. The operator should proceed sequentially in order to utilize the control chart[1].

1. Increment the sampling counter i by 1.
2. From now on, add to the sample x_i any observation x_{ij} of the quality characteristic measured from the output of the process, until the size of x_i reaches n. In the end, the set x_i is composed of n real valued numbers such that $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$. Note that the outputs should ideally be produced exactly at the same time, but, as it is most likely practically impossible, measures of consecutive outputs of the process may be added to the sample and will be treated as if they had been generated simultaneously by the process.
3. Compute a statistic y_i, which depends on the type of control chart, according to the function $y_i = f_{\mathcal{C}}(x_i)$. A statistic is a number computed from a sample that synthesizes one or several characteristics of a given sample (Coladarci et al., 2010).

[1] We suppose here that the snapshot sampling scheme is used.

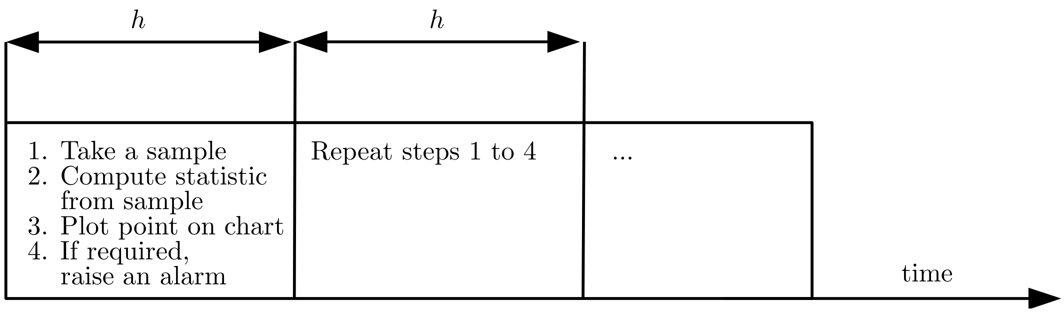


Figure 3.2: How does a control chart work ?

4. Plot the point $(i; y_i)$ on the control chart.

5. Raise a signal if $y_i \geq UCL$ or $y_i \leq LCL$.

6. Wait some time and go back to step 1. The time difference between two steps 1 must be h, where h is usually called the sampling interval.

This sequential procedure is summarized in four steps in Figure 3.2. From its definition, we see directly that control charts are tools that are quite simple to use. However, their behavior is controlled by four main parameters whose values must be carefully chosen in order to maximize the efficiency of the chart. These parameters are the sample size n, the sampling interval h, and the position of the control limits UCL and LCL. The values given to the parameters influence differently the behavior of the control chart. For example, a larger sample size generally makes it easier to detect small deviations.

The exact mechanisms of control charts depend on the existence of the control and warning limits (see Section 3.1.4), and vary from one control chart to another. The function f_C is usually fixed for a given control chart, and we discuss in Chapter 4 how the parameters can be determined in order to maximize the chart efficiency.

3.1.3 Performance analysis of control charts

An interesting parallel can be drawn between control charts and hypothesis testing. Actually, the control chart can be seen as a test of the hypothesis that the process is in an in-control state. The null hypothesis H_0 of a potential hypothesis test would be 'the process is in-control', while the alternative hypothesis H_1 would be 'the process is out-of-control'. If, from the analysis of the control chart, one concludes that the process is in-control, this is equivalent to failing to reject the null hypothesis. On the other hand, if the analysis of the control chart indicates that the process is out-of-control, this is equivalent, in terms of hypothesis testing, to rejecting the null hypothesis, and thus accepting the alternative hypothesis.

The parallel with hypothesis testing is useful for characterizing the performances of a control chart in terms of type I and type II errors. Indeed, concluding that the process is out-of-control, while it is actually in-control, is equivalent to a type I error in the hypothesis testing framework. Similarly, failing to detect that the process is actually out-of-control corresponds to a type II error. The type I and type II error probabilities, respectively denoted α and β, are thus natural measures of the performances of control charts. These measures give insight about how reliable the control chart is.

Another traditional measure to evaluate the performances of control charts is to use the concept of average run length (ARL). The ARL represents the average number of samples after which a signal will be raised by the control chart. The average run length is computed differently depending on whether we want the in-control ARL, denoted ARL_{ic}, or the out-of-control ARL, ARL_{oc}. The ARL values are given by

$$\text{ARL}_{\text{ic}} = \frac{1}{\alpha}, \tag{3.1}$$

$$\text{ARL}_{\text{oc}} = \frac{1}{1-\beta}, \tag{3.2}$$

where α and β represent the type I and type II errors, respectively. ARL_{ic} represents the average number of samples after which a signal will be raised by the chart, even though the process is in-control. ARL_{ic} thus indicates the average number of samples before a false alarm. A good control chart design will seek to maximize the value of ARL_{ic} in order to minimize the number of false alarms. On the other hand, the ARL_{oc} represents the average number of samples before a signal is raised by the chart when the process is out-of-control. In this case, a good design will try to minimize the value of ARL_{oc} in such a way that assignable causes are detected as quickly as possible.

The ARL is expressed in number of samples. Comparing two control charts, or different designs of a single control chart, with ARL makes no sense if their sampling intervals are different. The average time to signal (ATS) is a measure that expresses the average amount of time after which a signal will be raised by the chart. The ATS, both for IC and OC states, can be computed from the ARL. It has a similar interpretation, and allows to compare charts having different sampling intervals. The ATS is given by

$$\text{ATS} = \text{ARL} \times h. \tag{3.3}$$

In a slightly different way, the performances of control charts can also be assessed in monetary terms. Indeed, costs can usually be associated with most aspects of the process. For example, a cost can be defined when the process operates in the in-control and out-of-control states, the latter one being generally larger. Similarly, operations of investigating alarms, be they legitimate or not, can be priced by the company. Because the cost is what obviously matters most to the companies, the expected cost of operating the process is another good way to measure the performance of a control chart.

3.1.4 Refinements in control charts

In the beginning of this chapter, we have described how traditional control charts work. Other refinements can be implemented in order to better take into account the problem of interest. Some of them are discussed below.

One control limit

When a process deviates from its in-control values, there are two possible directions: the controlled quality characteristic can drift either to smaller or larger values. In cases where deviations are possible in one direction only or where one direction only is of interest, one possible refinement is to use one control limit, instead of two. Then, only deviations in the given direction can be detected. This refinement makes the control chart easier to implement and to analyze since irrelevant items are removed from it.

Warning control limits

Rather than removing control limits as in the previous case, another possible refinement would be to add other control limits, called 'warning control limits', somewhere between the central line and the upper and lower control limits. If one or a series of points fall between the warning control limits and the outermost limits, the process might be out-of-control. Some actions might then be taken. For example, we could increase the sampling frequency or the sample size to faster collect more information. This type of control limits usually increases the sensitivity of the chart, i.e. the warning limits allow the control chart to signal an out-of-control state more quickly. This is achieved at the expense of an increase in the number of false alarms. Another disadvantage of this type of chart is the increased effort required for its design since other control limits need to be determined.

Pattern analysis

Raising a signal when a point falls outside the control limits might not be enough to reduce the variability. Patterns in the plotted points are usually the sign that something goes wrong and should be corrected. The problem is then to recognize those interesting patterns that, when removed, would reduce the variability of the process. This is now a pattern recognition problem. Several rules of thumb were developed to take the patterns of the points in control charts into account. For example, two possible rules might be 'eight successive points fall above (or below) the central line', and 'a sequence of six points monotonically increasing (or decreasing)' (Montgomery, 2007).

Combining sensitizing rules

Traditionally, a signal is raised when one or more points fall outside the control limits UCL and LCL. The other rules using warning control limits and patter analysis are meant to boost the sensitivity of the chart so that the operators can react more quickly to an assignable cause that changed the process distribution. Another possible refinement is to combine several sensitizing rules, as those described previously. Combining different sensitizing rules usually decreases the type II error, but, on the contrary, increases the type I error. These rules must therefore be used with caution to avoid degrading the performances of the chart by an excessively high sensitivity. Furthermore, the combination of several rules generally makes the chart harder to use, understand and analyze.

3.2 Parametric control charts

We present in this section some control charts used to control the mean of a process and its variability around the mean. Usually, a control chart used to control the mean is associated with a control chart for the variability around the mean in order to detect different types of assignable causes. We will present both types here, although only the chart for the mean will be considered in the rest of this report.

The term parametric indicates that these control charts make assumptions on the distribution that generates the data, i.e. the distribution of the quality characteristic. These assumptions usually ease the design of the chart, but may be wrong in some circumstances. We will see in the next section how this type of assumption can be avoided.

3.2.1 The $\overline{x}$ chart: a control chart for the mean

The '$\overline{x}$ control chart' is a control chart that is traditionally used to control the mean of a process. The $\overline{x}$ chart keeps track, in a sense, of the variability of the process between consecutive samples, i.e. the variability of the process in the long term. It assumes that the quality characteristic whose mean we want to control is distributed according to a normal distribution with mean μ_{ic} and standard deviation σ_{ic}. Let $(x_{i1}, x_{i2}, \ldots, x_{in})$ be the ith sample of size n, then the statistic is computed according to

$$y_i = f_{\mathcal{C}}\left(x_i\right) = \overline{x}_i = \frac{x_{i1} + x_{i2} + \ldots + x_{in}}{n}, \tag{3.4}$$

which is distributed according to a normal distribution of mean μ_{ic} and standard deviation $\sigma_s^{\text{ic}} = \frac{\sigma_{\text{ic}}}{\sqrt{n}}$ (Montgomery, 2007). This is convenient since it permits to derive theoretically the value of α, corresponding to the type I error probability, for a given sample size n and given control limits UCL and LCL :

$$\alpha = \frac{1}{2}\left[1 + \text{erf}\left(\frac{\text{LCL} - \mu_{\text{ic}}}{\sigma_s^{\text{ic}}\sqrt{2}}\right)\right] + \frac{1}{2}\left[1 - \text{erf}\left(\frac{\text{UCL} - \mu_{\text{ic}}}{\sigma_s^{\text{ic}}\sqrt{2}}\right)\right], \tag{3.5}$$

where this expression uses the cumulative distribution function of the normal distribution to compute α.

The probability of type II errors can similarly be derived theoretically provided that the out-of-control mean μ_{oc} and standard deviation σ_{oc} are known, and that this OC state is unique. This means that, when the process goes out-of-control, the quality characteristic is distributed according to a normal distribution with mean μ_{oc} and standard deviation σ_{oc}. Under this assumption, the type II error for a given sample size n and given control limits UCL and LCL is :

$$\beta = \frac{1}{2}\left[1 + \text{erf}\left(\frac{\text{UCL} - \mu_{\text{oc}}}{\sigma_s^{\text{oc}}\sqrt{2}}\right)\right] - \frac{1}{2}\left[1 + \text{erf}\left(\frac{\text{LCL} - \mu_{\text{oc}}}{\sigma_s^{\text{oc}}\sqrt{2}}\right)\right], \tag{3.6}$$

where $\sigma_s^{\text{oc}} = \frac{\sigma_{\text{oc}}}{\sqrt{n}}$ is the standard deviation of a sample of size n drawn from a normal distribution of standard deviation σ_{oc}.

3.2.2 The R chart: a control chart for the variability

The 'R control chart' is used to control the variability range of the quality characteristic. Unlike the $\overline{x}$ chart, the R chart watches the variability of the process in a given sample, i.e. its role is to evaluate the instantaneous variability of the process. Given a sample $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$, the statistic of the R chart is computed as follows

$$y_i = f_{\mathcal{C}}(x_i) = R_i = \max_j x_{ij} - \min_j x_{ij}, \tag{3.7}$$

where R_i is called the range of the sample. The values of the mean and of the standard deviation of the statistic R have been well known for a long time. Assuming that all x_{ij} are independent identically distributed random variables, and that their cumulative distribution function is $F(x)$, it can be shown that the mean and the variance of R for a sample of size n is given by:

$$\mu_R = \int_{-\infty}^{+\infty} \left[1 - F(x')^n - \left(1 - F(x')\right)^n\right] dx', \tag{3.8}$$

$$\sigma_R^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{x'_n} \left[1 - F(x'_n)^n - \left(1 - F(x'_1)\right)^n - \left(F(x'_n) - F(x'_1)\right)^n\right] dx'_1 dx'_n - \mu_R^2, \tag{3.9}$$

where $R = x'_n - x'_1$, i.e. x'_n represents the largest variable in the sample and x'_1 the smallest (Tippett, 1925; Mardia, 1965). These equations are somewhat complicated, but some authors tabulated the values of μ_R and σ_R^2 for several distributions and several values of n, see for example the work of Tippett (1925) or Montgomery (2007). Of course, computing the values of these integrals is no longer a major difficulty with modern computers.

Similarly to the $\overline{x}$ chart, these values can be used to determine the CL, LCL and UCL for the R chart. One important difference is that, in this case, the distribution of the statistic R is not normal, even if the x's are normally distributed (Montgomery, 2007). The computed limits are thus approximations of the real limits, and the practical results will reveal discrepancies with the theory.

3.3 Nonparametric control charts

The basic assumption of the $\overline{x}$ chart is that the distribution of the quality characteristic is normal. If we have evidence that this is not the case, and if the real distribution is known, it is possible to derive the $\overline{x}$ and R charts for the new distribution in the same way as for the normal distribution. However, when the real distribution is unknown, or when it is simply too difficult to derive the optimal values for the control limits, it might be preferable to use nonparametric control charts (Woodall, 2000). Some practitioners think that the central limit theorem renders the development of nonparametric charts groundless (Chakraborti et al., 2001). However, this is not true for control charts based on non-average values, and for those charts that are used with samples of size one.

Nonparametric charts are formally defined as those control charts for which the distribution of the run length when the process is in-control is the same, whatever the continuous distribution of the quality characteristic (Chakraborti et al., 2001). Nonparametric charts, also called distribution-free charts, generally make weaker probabilistic assumptions about the distribution of the quality characteristic than parametric charts. For example, no special shape is assumed, but some properties, like symmetry of the distribution, may be required. Furthermore, nonparametric charts may control a different statistic than traditional charts. They are traditionally used to control the median or another percentile of the distribution. One advantage of using the median as statistic of a sample is that it is always defined for any distribution, unlike the mean (Gibbons and Chakraborti, 2011).

In the remainder of the section, we will describe two famous nonparametric charts that can favorably replace parametric charts in some situations. Note that we consider here non-parametric charts that have control limits in only one direction, the upper direction, and that we denote by k the upper control limit, instead of UCL.

3.3.1 The SN chart

The 'SN chart' is based on the sign test, which is probably the simplest of nonparametric tests (Gibbons and Chakraborti, 2011). This test can be used to check statistical hypotheses about any quantile, and in particular the median, of any continuous distribution (Chakraborti et al., 2001). This test has a large variety of applications since it may be applied even if the distribution is not symmetric.

Presentation of the statistic

In the case of the median, the statistic of the SN chart is computed as follows

$$y_i = f_{\mathcal{C}}(x_i) = \mathrm{SN}_i^{\theta_{\mathrm{ic}}} = \sum_{j=1}^{n} \mathrm{sign}\left(x_{ij} - \theta_{\mathrm{ic}}\right), \tag{3.10}$$

where θ_{ic} is the in-control median of the quality characteristic, and sign$(\cdot)$ is the sign function that returns -1, 0, or 1, when its argument is stricly less, equal, or greater than 0, respectively. If any other quantile other than the median is to be controlled, the value of θ_{ic} has to be changed in this formula for the in-control value of the quantile of interest.

Distribution of the statistic

There exists a relationship between the sign statistic $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ and the 'traditional' sign statistic defined by $\mathrm{K}_i^{\theta_{\mathrm{ic}}} = \sum_{j=1}^{n} \mathbf{1}_{\mathbb{R}_0^+}\left(x_{ij} - \theta_{\mathrm{ic}}\right)$, where $\mathbf{1}_{\mathbb{R}_0^+}(\cdot)$ is an indicator function that returns 1 if its argument is strictly positive, and 0 otherwise. The $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ and $\mathrm{K}_i^{\theta_{\mathrm{ic}}}$ statistics obey the following linear relation

$$2\mathrm{K}_i^{\theta_{\mathrm{ic}}} = \mathrm{SN}_i^{\theta_{\mathrm{ic}}} + n, \tag{3.11}$$

that permits to compute the distribution of $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ from that of $\mathrm{K}_i^{\theta_{\mathrm{ic}}}$. When the process is in-control, the statistic $\mathrm{K}_i^{\theta_{\mathrm{ic}}}$ is distributed according to a binomial distribution $B\left(n, 0.5\right)$ so that

$$P_{\mathrm{K}_i^{\theta_{\mathrm{ic}}}}\left[z|\,\mathrm{IC}\right] = \binom{n}{z}(0.5)^z(0.5)^{n-z}, \tag{3.12}$$

where $P_{\mathrm{K}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}]$ represents the probability of $\mathrm{K}_i^{\theta_{\mathrm{ic}}}$ being equal to z when the process is in-control.

When the process is in-control, the distribution of the statistic $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ to control the median can be computed from (3.12). It is given by

$$P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}] = P_{\mathrm{K}_i^{\theta_{\mathrm{ic}}}}\left[\frac{z+n}{2}\middle|\,\mathrm{IC}\right] = \frac{n!}{\left(\frac{n+z}{2}\right)!\left(\frac{n-z}{2}\right)!}(0.5)^{\left(\frac{n+z}{2}\right)}(0.5)^{\left(\frac{n-z}{2}\right)}, \tag{3.13}$$

where Equation (3.13) is valid as long as both n and z have the same parity. Note that, due to the definition of SN, the distribution (3.13) is valid for any statistic $\mathrm{SN}_i^{\theta'}$ as long as the median of the distribution of the observations x_{ij} is θ'. In particular, this is true for θ_{oc} such that $P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}] = P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}}[z|\,\mathrm{OC}]$.

Note that this distribution is correct as long as the probability of having a x_{ij} being exactly equal to the median is null. In theory, this is a valid assumption for continuous distributions, but it might not be true in practical applications. In this case, some workarounds need to be found to ensure that the distribution is still valid, even if some values in the sample are equal to the median. Such strategies are not described here, but some of them can be found in the book of Gibbons and Chakraborti (2011).

Type I and type II error probabilities

The probability of type I errors of the sign statistic $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ for a control limit k can finally be obtained by summing over the possible values that $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ can take:

$$\alpha = \sum_{z=k}^{n} P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}], \tag{3.14}$$

where the sum is done over all z that have the same parity as n.

The type II error probability is unfortunately a bit more complicated to compute. More on this matter can be found in Section 5.1.

3.3.2 The SR chart

The 'SR chart' is based on the so-called Wilcoxon signed-rank statistic and is used to detect drifts of a given location parameter, for instance the median or another quantile, from its in-control value (Gibbons and Chakraborti, 2011). Unlike the SN chart, SR charts require that the distribution of the quality characteristic be symmetric.

Presentation of the statistic

When we are interested in controlling the deviations of the quality characteristic from its median θ_{ic}, the statistic of the SR chart (Chakraborti et al., 2011), called the Wilcoxon signed-rank statistic, is given by

$$y_i = f_{\mathcal{C}}(x_i) = \mathrm{SR}_i^{\theta_{\mathrm{ic}}} = \sum_{j=1}^{n} \mathrm{sign}(x_{ij} - \theta_{\mathrm{ic}}) R_{ij}, \tag{3.15}$$

where R_{ij} is the rank of $x_{ij} - \theta_{\mathrm{ic}}$ when the set $(|x_{i1} - \theta_{\mathrm{ic}}|, |x_{i1} - \theta_{\mathrm{ic}}|, \ldots, |x_{in} - \theta_{\mathrm{ic}}|)$ is sorted in ascending order. Similarly to the SN chart, we assume all x_{ij} to be different from the median θ_{ic}.

Distribution of the statistic

As for the SN chart, the $\mathrm{SR}_i^{\theta_{\mathrm{ic}}}$ statistic depends linearly on the simpler statistic $\mathrm{W}_i^{\theta_{\mathrm{ic}}} = \sum_{j=1}^{n} \mathbf{1}_{\mathbb{R}_0^+}(x_{ij} - \theta_{\mathrm{ic}}) R_{ij}$ through the relation

$$\mathrm{SR}_i^{\theta_{\mathrm{ic}}} = 2\mathrm{W}_i^{\theta_{\mathrm{ic}}} - \frac{n(n+1)}{2}, \tag{3.16}$$

see (Chakraborti et al., 2011). When the process is in-control, the statistic $\mathrm{W}_i^{\theta_{\mathrm{ic}}}$ is distributed according to a recursively determinable probability distribution given by

$$P_{\mathrm{W}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}] = \frac{u_n(z)}{2^n}, \tag{3.17}$$

where $u_n(z)$ represents the number of vectors c composed of zeros and ones such that the dot product of c with the vector composed of the integers $\{1, \ldots, n\}$ is equal to z (Wilcoxon et al., 1970; Gibbons and Chakraborti, 2011). The value of $u_n(z)$ can be obtained through the recursive formula

$$u_n(z) = u_{n-1}(z - n) + u_{n-1}(z), \tag{3.18}$$

which can be initialized for $n = 2$ with $u_2(0) = u_2(1) = u_2(2) = u_2(3) = 1$.

In the end, the probability distribution of $\mathrm{SR}_i^{\theta_{\mathrm{ic}}}$ can be found from that of $\mathrm{W}_i^{\theta_{\mathrm{ic}}}$ with a simple transformation

$$P_{\mathrm{SR}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}] = P_{\mathrm{W}_i^{\theta_{\mathrm{ic}}}}\left[\frac{z}{2} + \frac{n(n+1)}{4}\middle|\,\mathrm{IC}\right]. \tag{3.19}$$

Note that, due to the definition of the Wilcoxon signed rank statistic, the distribution (3.19) is valid for any $\mathrm{SR}_i^{\theta'}$ as long as the median of the symmetric distribution of the observations x_{ij} is θ'. In particular, this is true for θ_{oc} such that $P_{\mathrm{SR}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}] = P_{\mathrm{SR}_i^{\theta_{\mathrm{oc}}}}[z|\,\mathrm{OC}]$.

Type I and type II error probabilities

The probability of type I errors for a control limit k is given by

$$\alpha = \sum_{z=k}^{\frac{n(n+1)}{2}} P_{\mathrm{SR}_i^{\theta_{\mathrm{ic}}}}[z|\,\mathrm{IC}], \tag{3.20}$$

where the sum is done over the z having the same parity as $\frac{n(n+1)}{2}$.

The type II error probability is slightly more complicated to compute. We refer the reader to Section 5.1 for more information on this issue.

Chapter 4

Designing control charts

In this chapter, we describe some methods that can be used to determine all the parameters whose values must be known in order to practically implement a control chart. Among these parameters, there are characteristics of the distribution of the quality characteristic, as well as the three fundamental parameters of the control charts: the sample size, the sampling interval and the control limit. We first explain in Section 4.1 how some characteristics of the distribution of the quality characteristic, such as the mean, the standard deviation and the median, can be estimated from the process when they are unknown. Section 4.2 then describes the most common methods that are used to determine the sample size, the sampling interval and the control limits for a given control chart.

Note that, from now on, we will only consider control charts for the mean or the median with one control limit only, the upper control limit, denoted by k in the remainder of the document. Indeed, we will assume that deviations of the quality characteristic towards larger values are the only deviations that are harmful to the process.

4.1 Characteristics of the distribution of the quality characteristic

In general, the mean μ, the standard deviation σ and the median θ, describing the distribution of the quality characteristic, are unknown. However, these characteristics must usually be known in advance in order to determine the parameters k, n and h for a given control chart. In that case, only estimated values $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\theta}$ can be used to implement the chart. Typically, the procedure to estimate those parameters is to collect a certain number m of samples, each containing n measures of the quality characteristic, and to infer estimates of the characteristics from the collected data. This is usually done during a so-called phase I, whose goal is to first bring the process in-control and to estimate μ, σ and θ. The theory of estimation in statistics provides the useful tools to find $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\theta}$. The estimators of the mean and of the standard deviation are used in both $\overline{x}$ and R charts. As for the median, it is mainly used in nonparametric charts such as the SN and the SR chart.

An estimator of the mean can be obtained by taking the empirical average of the empirical averages of each sample, i.e.

$$\tilde{\mu} = \frac{\overline{x}_1 + \overline{x}_2 + \ldots + \overline{x}_m}{m}, \tag{4.1}$$

with

$$\overline{x}_i = \frac{x_{i1} + x_{i2} + \ldots + x_{in}}{n}. \tag{4.2}$$

The estimator $\tilde{\sigma}$ of the standard deviation is only slightly more tedious to compute:

$$\tilde{\sigma} = \frac{s_1 + s_2 + \ldots + s_m}{m} \left(\sqrt{\frac{2}{n-1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \right)^{-1}, \tag{4.3}$$

with

$$s_i = \sqrt{\frac{\sum_{j=1}^{n} \left(x_{ij} - \overline{x_i}\right)^2}{n-1}}, \tag{4.4}$$

where the s_i represent the biased estimators of the standard deviations of each sample, and the second factor of Equation (4.3) is a correction factor.

The median θ is the last characteristic of the distribution that we may want to approximate. The median estimator is obtained directly by taking the mean over the samples of the medians computed for each sample, i.e.

$$\tilde{\theta} = \frac{\theta'_1 + \theta'_2 + \ldots + \theta'_m}{m}, \tag{4.5}$$

where

$$\theta'_i = \text{median}\left(x_{i1} + x_{i2} + \ldots + x_{in}\right). \tag{4.6}$$

Generally, the number m of samples used to estimate the parameters of the distribution is around 20 to 25, and the number n of measures in each sample is small, often of the order of 5 (Montgomery, 2007). These formulas can be used during the phase I to find an estimate of the characteristics of the distribution of the quality characteristic of interest. It is to be noted that, though standard values might seem preferable to estimated ones, standard values must be used with caution, since there is no guarantee that the standard values actually reflect the in-control state of the process.

4.2 Determining the main parameters of the control chart

Once the mean, the standard deviation and the median of the distribution of the quality characteristic are known, we can start looking into the problem of determining the values of the parameters k, n and h. A meaningful solution to this matter is usually impossible to find without additional information about the distribution of the quality characteristic, such as its shape, as well as economic information about the process itself (Montgomery, 2007). We present here two methods, namely the heuristic method and the statistical design method, that only exploit information from the distribution of the quality characteristic. The other two described methods, namely economic and economic statistical design, make use of economic information as well.

Since the focus of this report is the control of the mean/median of the process, we will focus on these aspects in the design of the control charts, and will thus omit the details regarding the R charts and related.

Note that three of the methods that we describe will exhibit a direct dependence on the probabilities of type I and type II errors, respectively denoted α and β. These probabilities depend themselves on the parameters (k, n, h) of the chart. We will emphasize this dependency by writing $\alpha(k, n, h)$ and $\beta(k, n, h)$. Even if the probabilities of type I and type II errors of the studied control charts do not depend on the sampling interval h, it might be that they depend on it for some other charts. For this reason, we explicitly allow α and β to depend on the three design parameters.

4.2.1 Heuristics

The historically first, simplest, and easiest way to determine the values of the parameters k, n and h is to use heuristic methods that have proved to work acceptably well in practice. These heuristics were first proposed by Shewhart (1931) for the $\overline{x}$ chart and are described here in this context. They consist mainly in using rules of thumb to determine the values of the parameters.

The $\overline{x}$ chart is meant to detect relatively large shifts in the process mean. In that case, rather small sample sizes of the order of $n = 5$ usually produce good results. When the shifts to detect are smaller, larger samples sizes are required, and the number of observations in each sample might be increased to $n = 15$ or 25, and even more (Montgomery, 2007). Choosing h and n is typically a problem of effort allocation. Indeed, the company usually has a limited amount of resources that it is willing to devote to the sampling procedure. The two possible choices are either to take often small samples, or to take less larger samples. There is no a priori best decision, but the current practice in industry is biased towards smaller samples taken at shorter time intervals (Montgomery, 2007). The final decision will also depend on the process itself. For example, if the process produces many units per hour, it might be better to take samples very often. Indeed, if the sampling interval is too long, the process will produce many defective products before the shift is detected.

With regard to the control limits, a common practice in the design of $\overline{x}$ charts is to use 3-sigma limits that consist in placing the control limits $3\sigma_s$ away from the center line. The value of k is thus given by $k = \mathrm{CL} + 3\sigma_s$. Note that 'sigma' here refers to the standard deviation of the statistic $\sigma_s = \frac{\sigma}{\sqrt{n}}$, and not to the standard deviation σ of the quality characteristic. Multiplying the standard deviation by 3 is the standard design choice, but other multipliers can be used if additional information is available. For example, if we know that false alarms are really expensive to investigate, the designer might choose to push forward the limits and to place the control limit at 3.5-sigma instead. Similarly, if the search for assignable causes is very cheap, it might be better to carry out a lot of them so that the process remains out-of-control as short a time period as possible. This might result in narrower control limits placed at 2.5-sigma, for example.

When the control limits are computed by adding to the center line a multiple of the standard deviation of the sample, the limits are called r-sigma limits. Another way to specify the limits, which in this case are called probability limits, is to choose a multiplier that induces a given probability of type I errors. In this case, the type I error probability, i.e. the value of α, is specified, and the value of r is then calculated to achieve the desired type I error probability. Note that, when there are both UCL and LCL, the traditional 3-sigma limits induce a probability of type I error of 0.0027, which has to be compared with the traditional $\alpha = 0.002$ that the advocates of the probability limits use. The difference between both approaches is thus rather tiny.

Note that both the sigma and the probability limits are computed with the underlying assumption that the distribution of the quality characteristic follows a normal distribution. If it is not the case, the computed limits do not offer any guarantees on the probabilities of type I and type II errors.

4.2.2 Economic design

The economic design approach dates back to the 1950s (Duncan, 1956). The method consists in finding the main parameters of the chart (k, n, h) such that the expected cost of operating the process is minimized. We will describe here the cost model of Duncan (1956). Other cost functions have later been developed, see e.g. the survey papers of Montgomery (1980) and Vance (1983), but their thorough description is beyond the scope of this report.

Duncan's model is based on the assumption that we can, in some way or another, price different aspects of the process, such as the cost per time unit of operating in-control, the cost per time unit of being out-of-control, and the cost of evaluating true and false alarms. In his framework, Duncan models the expected cost of operating the process per time unit. This cost per time unit is obtained by dividing the expected cost of a cycle by the expected cycle time. In this context, a cycle is defined by four periods of time, which are: (1) the in-control period, (2) the time to signal the assignable cause, (3) the time to take and examine a sample, and (4) the time to identify and correct the assignable cause. The different periods are illustrated in the Figure 4.1. An additional assumption of this model is that no assignable cause can occur during the sampling process.

When the process is first assumed to be in-control, the expected cycle time $T_{exp}(k, n, h)$ is composed of four components.

1. The mean time before an assignable cause occurs. The assignable causes are assumed to follow a Poisson process, and the time difference between two causes thus follows an exponential distribution of mean $\frac{1}{\lambda}$, the mean time before an assignable cause is thus $\frac{1}{\lambda}$.
2. The adjusted average time to signal (AATS) (Faraz et al., 2013) is the expected amount of time between the occurrence of an assignable cause and a signal raised by the control chart. It is composed of two components:

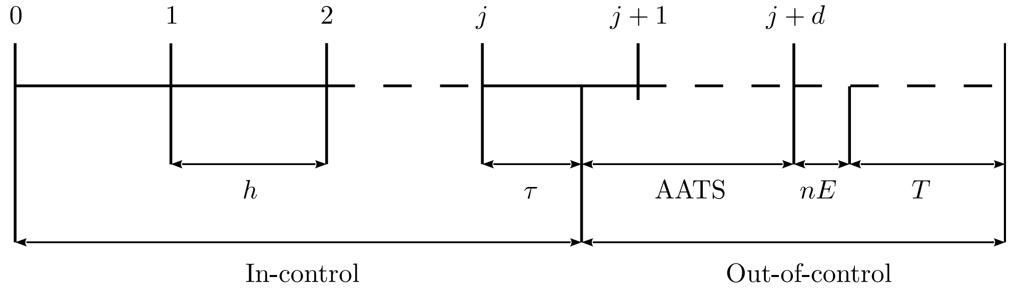


Figure 4.1: Expected cycle time $T_{exp}(k, n, h)$ and its components for the economic design of control charts.

(a) the expected time between the occurrence of an assignable cause and the next sample, which is given by $h - \tau(h)$, where $\tau(h) = \frac{1-\exp(-\lambda h)(1+\lambda h)}{\lambda(1-\exp(-\lambda h))}$ is the average amount of time after which an assignable cause occurs given that it occurs between samples j and $j+1$;

(b) the expected amount of time, from the first sample after the occurrence of the cause, required by the chart to signal an out-of-control state, which is given by $h\,(\text{ARL}_{\text{oc}}(k, n, h) - 1)$ where $\text{ARL}_{\text{oc}}(k, n, h) = \frac{1}{1-\beta(k,n,h)}$.

3. The expected time to take a sample of n measurements and to interpret the results. The expected time to take and analyze one observation is denoted E, and the total expected time for the entirety of a sample of size n is thus nE.

4. The expected time to identify and correct the assignable cause, denoted by T.

In the end, the expected cycle time is given by

$$\begin{aligned} T_{exp}(k, n, h) &= \frac{1}{\lambda} + h - \tau(h) + h\,(\text{ARL}_{\text{oc}}(k, n, h) - 1) + nE + T, \\ &= \frac{1}{\lambda} - \tau(h) + h\text{ARL}_{\text{oc}}(k, n, h) + nE + T, \\ &= \frac{1}{\lambda} - \tau(h) + \frac{h}{1 - \beta(k, n, h)} + nE + T. \end{aligned} \tag{4.7}$$

Duncan's model finds the optimal values of the parameters by minimizing the expected cost per time unit of operating the process. We thus still need to compute the estimated cost of the cycle. The expected cost $Q_{exp}(k, n, h)$ of the cycle is composed of three terms.

1. The expected cost of operating the process, when it is both in-control and out-of-control. This cost is given by $\frac{C_0}{\lambda} + C_1\,(h\text{ARL}_{\text{oc}}(k, n, h) - \tau(h) + nE + T)$, where C_0 represents the cost per time unit when the process is in-control, and C_1 denotes the cost per unit of time when the process is out-of-control.

2. The expected cost of investigating both true and false alarms. This cost is given by $W\alpha(k, n, h)\frac{\exp(-\lambda h)}{1-\exp(-\lambda h)} + Y$, where W is the cost of investigating false alarms, $\alpha(k, n, h)$ is the type I error probability, and Y is the cost of investigating a true alarm and repairing the process.

3. The expected cost of sampling per cycle. This cost is given by

$$\frac{S}{h} T_{exp}(k,n,h) = \frac{S}{h}\left(\frac{1}{\lambda} - \tau(h) + \frac{h}{1-\beta(k,n,h)} + nE + T\right),$$

   where S denotes the sampling cost per sample.

When all the terms are put together, the expected cost of a cycle is given by

$$\begin{aligned} Q_{exp}(k,n,h) = & \frac{C_0}{\lambda} + C_1\left(h\mathrm{ARL}_{\mathrm{oc}}(k,n,h) - \tau(h) + nE + T\right) \\ & + W\alpha(k,n,h)\frac{\exp(-\lambda h)}{1-\exp(-\lambda h)} + Y \\ & + \frac{S}{h}\left(\frac{1}{\lambda} - \tau(h) + \frac{h}{1-\beta(k,n,h)} + nE + T\right), \end{aligned} \tag{4.8}$$

and the expected cost per time unit $C_{exp}(k,n,h)$ is finally obtained by dividing the cost of a cycle by the expected duration of the cycle

$$C_{exp}(k,n,h) = \frac{Q_{exp}(k,n,h)}{T_{exp}(k,n,h)}. \tag{4.9}$$

Economic design consists now in finding the values of the parameters (k^*, n^*, h^*) that minimize the expected cost per time unit $C_{exp}(k,n,h)$. This can be done by formulating an optimization problem to minimize the nonlinear mixed-integer cost function $C_{exp}(k,n,h)$. The optimization problem is as follows:

$$\begin{aligned} (k^*, n^*, h^*) = \arg\min_{k,n,h} \quad & C_{exp}(k,n,h) \\ \text{subject to} \quad & h > nE, \\ & k > 0, \\ & n \in \mathbb{N}_0^+, \\ & h > 0, \end{aligned} \tag{4.10}$$

where the constraint $h > nE$ is added to make sure that the sampling interval is not shorter than the amount of time needed to actually take and analyze a sample.

This optimization problem is rather hard to solve because it is both nonlinear and mixed-integer. We cannot expect to always find the infimum of the function. However, a local minimum will in general suffice.

4.2.3 Statistical design

Statistical design is a method that has been created to alleviate some of the identified problems of economic design that was depicted in the previous section. Woodall (1985) identified two major problems of economic design. First, Woodall (1985) argues that the values of the parameters obtained through economic design are very sensitive to the shift in the mean. If the shift that actually occurs when the process goes out-of-control is a bit far from the expected shift, the design parameters might not be optimal at all. Secondly, economic design considers only false alarms through their total cost which is summed to other costs, and thus, in a sense, averaged. This can result in a very small $\mathrm{ARL}_{\mathrm{ic}}$, and thus in an unreasonable number of false alarms, that induces more variability and reduces the faith in the control procedure (Woodall, 1985).

Woodall (1985) proposed a new design method, called 'statistical design', that is based on the fact that the in-control ARL should ideally be lower bounded by some value, say L_{ic}, and that, similarly, the out-of-control ARL should be upper bounded by L_{oc}. In mathematical terms, these conditions are expressed as

$$\mathrm{ARL}_{\mathrm{ic}}(k,n,h) \geq L_{\mathrm{ic}}, \quad (4.11)$$

$$\mathrm{ARL}_{\mathrm{oc}}(k,n,h) \leq L_{\mathrm{oc}}. \quad (4.12)$$

Because of the existing relationship between average run length and type I and type II error probabilities, the bounds (4.11) and (4.12) can be reformulated as

$$\alpha(k,n,h) \leq \alpha_0 = \frac{1}{L_{\mathrm{ic}}}, \quad (4.13)$$

$$\beta(k,n,h) \leq \beta_0 = 1 - \frac{1}{L_{\mathrm{oc}}}. \quad (4.14)$$

This last formulation highlights the fact that these bounds are heavily dependent on the (assumed) distribution of the quality characteristic. One must thus be careful when using such design criteria because a mistake in the assumed distribution of the characteristic may have a great impact on the performance of the chart.

The previous bounds can usually not be used as is to determine the value of h. A simple workaround would consist in adding a constraint of the form

$$h\mathrm{ARL}_{\mathrm{oc}}(k,n,h) \leq \mathrm{ATS}_0, \quad (4.15)$$

where ATS_0 is the desired average time to signal of the chart when the process goes out-of-control.

Designing a control chart according to the statistical design method thus consists in choosing appropriate values for k, n, and h, such that the above equations are satisfied.

4.2.4 Economic statistical design

The last design method that we present here combines the ideas of both economic and statistical design (Saniga, 1989). Indeed, both methods are not exempt from drawbacks. The flaws of economic design have been identified by Woodall (1985) who acknowledges that statistical design has some shortcomings as well. One of them being that statistical design is not optimal in terms of operating costs (Woodall, 1985).

Saniga (1989) proposed an approach, called 'economic statistical design' (ESD), that combines the ideas of economic and statistical design in a single method. More specifically, economic statistical design consists in minimizing the expected operating cost per time unit, such as in economic design, subject to additional statistical constraints, such as in statistical design. This new problem can be formulated as

$$\begin{aligned}(k^*, n^*, h^*) = \arg\min_{k,n,h} \quad & C_{exp}(k, n, h) \\ \text{subject to} \quad & h > nE, \\ & \alpha(k, n, h) \leq \alpha_0, \\ & \beta(k, n, h) \leq \beta_0, \\ & k > 0, \\ & n \in \mathbb{N}_0^+, \\ & h > 0,\end{aligned} \tag{4.16}$$

which is a nonlinear mixed-integer optimization problem.

The advantage of economic statistical design is that its solution is actually a minimal cost solution for which statistical constraints are indeed satisfied. ESD is thus a method that yields values of the parameters k, n, and h that are more robust than the previous ones. The problem of finding an optimal solution to (4.16) remains, since (4.16) is mixed-integer and, most likely, non-convex.

Chapter 5

Economic statistical design of nonparametric control charts

This chapter focuses on the description of the economic statistical design (ESD) of nonparametric control charts, which is the main topic of this report.

Combining ESD and nonparametric charts is not difficult. It suffices to use, in the ESD problem formulation described in (4.16), the functions $\alpha(k, n, h)$ and $\beta(k, n, h)$ computing the probabilities of the type I and type II errors for the nonparametric chart that we are planning to use. The main difficulty consists in determining those α and β functions.

Note that the dependency of $\alpha(k, n, h)$ and $\beta(k, n, h)$ on the parameters of the chart is made implicit here by simply writing α and β.

5.1 Simple approach to nonparametric economic statistical design

We describe now the simple approach that we used to combine economic statistical design and nonparametric charts. The two nonparametric charts that we considered in this work share the same limitation: it is hard to determine the probability of type II errors for a given control limit. We show in this section how we got round that difficulty.

It is not possible to find exact expressions for both α and β in the case of nonparametric charts. Indeed, the available in-control and out-of-control distributions characterize different random variables. Furthermore, the simplest change of variables from the space of one random variable to the other is not exact and only gives bounds on the statistics. In this work, we compute explicitly α and we use the variable bounds to show that β can be upper bounded by a function of α. Note that the opposite, i.e. computing exactly the value of β and bounding α, is possible as well.

5.1.1 Type II error probability for the SN chart

In the case of the SN chart, the type I error probability for a given control limit k' can easily be computed with Equation (3.14) as

$$\alpha = \sum_{z=k'}^{n} P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}} \left[z | \, \mathrm{IC} \right], \tag{5.1}$$

and the type II error probability corresponding to the same control limit k' is given by

$$\beta = \sum_{z=-n}^{k'-1} P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}} \left[z | \, \mathrm{OC} \right], \tag{5.2}$$

where $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ is the statistic computed with in-control median θ_{ic}. Unfortunately, the discrete probability distribution $P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}} \left[z | \, \mathrm{OC} \right]$ is unknown, and there is, to our knowledge, no way to characterize it. This renders Equation (5.2) impossible to use, and other solutions have to be found to compute the β corresponding to control limit k'.

One possible workaround to compute the type II error probability is to use the out-of-control distribution of the statistic $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$. This distribution is known and is given by

$$P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}} \left[z | \, \mathrm{OC} \right] = \frac{n!}{\left(\frac{n+z}{2}\right)! \left(\frac{n-z}{2}\right)!} (0.5)^{\left(\frac{n+z}{2}\right)} (0.5)^{\left(\frac{n-z}{2}\right)}, \tag{5.3}$$

which is analogous to the probability distribution (3.13) in the in-control case. Using the distribution (5.3), the type II error probability can be computed for a given control limit k'' with

$$\beta = \sum_{z=-n}^{k''-1} P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}} \left[z | \, \mathrm{OC} \right]. \tag{5.4}$$

The difference between Equations (5.2) and (5.4) is that the distribution in Equation (5.4) is known, and can be computed. However, the problem with using the latter equation is that it is defined for a random variable, namely $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$, that is different from the one used in Equation (5.1), namely $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$. Because the random variables are different in both equations, a control limit for $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ does not correspond to the same control limit for $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$. In other words, a single control limit cannot be used for both Equations (5.1) and (5.4). A possible solution to this issue is to find a function T that maps one value of $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ to its corresponding $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$, i.e.

$$T\left(\mathrm{SN}_i^{\theta_{\mathrm{ic}}}\right) = \mathrm{SN}_i^{\theta_{\mathrm{oc}}}. \tag{5.5}$$

The function T corresponds in a sense to a change of variable that transforms a value from the space of $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ into the 'corresponding' value in the space of $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$. Because control limits are special values that the statistics can take, the function T can be used on the control limits as well to map a control limit k in one space to the other. Using this change of variable, the type I and type II error probabilities for a given control limit k' can be written in terms of known distributions depending on a single variable, namely k', such that

$$\alpha = \sum_{z=k'}^{n} P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}} \left[z | \, \mathrm{IC} \right], \tag{5.6}$$

$$\beta = \sum_{z=-n}^{k''-1} P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}} \left[z | \, \mathrm{OC} \right], \tag{5.7}$$

where $k'' = T\left(k'\right)$ is the control limit corresponding to k' in the space of $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$.

Unfortunately, it is not possible to find a function that gives precisely $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$ in terms of $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$ in general because the mapping depends on the sample and not only on $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$. This is due to the definition of the SN statistic that transforms the input space of the observations into a discrete space where the scale of the data does not play a role anymore. However, it is possible to find a relation of the form

$$\mathrm{SN}_i^{\theta_{\mathrm{oc}}} \leq \mathrm{SN}_i^{\theta_{\mathrm{ic}}}, \tag{5.8}$$

indicating that, in the worst case, $\mathrm{SN}_i^{\theta_{\mathrm{oc}}}$ is equal to $\mathrm{SN}_i^{\theta_{\mathrm{ic}}}$. The demonstration of this inequality is given at the end of this section.

Introducing this inequality in the previous equations, we have

$$\beta = \sum_{z=-n}^{k''-1} P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}} \left[z | \, \mathrm{OC} \right] \leq \sum_{z=-n}^{k'-1} P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}} \left[z | \, \mathrm{OC} \right], \tag{5.9}$$

since $k'' \leq k'$, and, given that $P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}} \left[z | \, \mathrm{IC} \right] = P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}} \left[z | \, \mathrm{OC} \right]$, we have

$$\sum_{z=-n}^{k'-1} P_{\mathrm{SN}_i^{\theta_{\mathrm{oc}}}} \left[z | \, \mathrm{OC} \right] = \sum_{z=-n}^{k'-1} P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}} \left[z | \, \mathrm{IC} \right] \tag{5.10}$$

$$= 1 - \sum_{z=k'}^{n} P_{\mathrm{SN}_i^{\theta_{\mathrm{ic}}}} \left[z | \, \mathrm{IC} \right] \tag{5.11}$$

$$= 1 - \alpha, \tag{5.12}$$

which, when combined with (5.9), gives

$$\beta \leq 1 - \alpha. \tag{5.13}$$

Although we cannot compute the value of β exactly for a given control limit, Equation (5.13) indicates that it is possible to upper bound it by $1 - \alpha$. The bound is quite loose, and tighter bounds would of course be preferable. However, this bound is enough to be used within economic statistical design.

Although the demonstration is quite trivial, we now demonstrate why Equation (5.8) always holds. Let $(x_{i1}, x_{i2}, \ldots, x_{in})$ be a random sample drawn from any distribution, and let θ_{ic} and θ_{oc} be two values taken in the space of the observations such that $\theta_{\mathrm{ic}} \leq \theta_{\mathrm{oc}}$. Remember that the probability for an observation x_{ij} to be equal to either θ_{ic} or θ_{oc} is null. Let us split the sample into three subsets: S_i, I_i and G_i. These subsets are defined by

$$\begin{aligned} S_i &= \{x_{ij} \mid \forall j : x_{ij} < \theta_{\text{ic}} \;\&\; x_{ij} < \theta_{\text{oc}}\}, \\ I_i &= \{x_{ij} \mid \forall j : x_{ij} > \theta_{\text{ic}} \;\&\; x_{ij} < \theta_{\text{oc}}\}, \\ G_i &= \{x_{ij} \mid \forall j : x_{ij} > \theta_{\text{ic}} \;\&\; x_{ij} > \theta_{\text{oc}}\}. \end{aligned}$$

Then, the statistics $\text{SN}_i^{\theta_{\text{ic}}}$ and $\text{SN}_i^{\theta_{\text{oc}}}$ are respectively given by

$$\text{SN}_i^{\theta_{\text{ic}}} = \sum_{j=1}^{n} \text{sign}\,(x_{ij} - \theta_{\text{ic}}) = |G_i| + |I_i| - |S_i|, \tag{5.14}$$

$$\text{SN}_i^{\theta_{\text{oc}}} = \sum_{j=1}^{n} \text{sign}\,(x_{ij} - \theta_{\text{oc}}) = |G_i| - |I_i| - |S_i|, \tag{5.15}$$

where the operator $|\cdot|$ represents the cardinality of the set given in argument. Introducing Equations (5.14) and (5.15) into Equation (5.8) proves the latter directly since the cardinality of a set is always greater or equal to 0.

If the type II error is deemed more important than the type I error, we can still compute its exact probability and bound the value of α instead, but this requires to use the statistic $\text{SN}_i^{\theta_{\text{oc}}}$ instead of $\text{SN}_i^{\theta_{\text{ic}}}$ to perform the tests. Actually, this amounts to creating an 'inverse' statistical hypothesis test with a null hypothesis being: 'the process is out-of-control'.

5.1.2 Type II error probability for the SR chart

Similarly to the SN chart, the known in-control and out-of-control distributions of the SR chart characterize different statistics, which are computed differently. For the same reasons, it is not possible to find a single control limit for both α and β depending on the known distributions. However, the procedure that we applied for the SN chart can be used for the SR chart as well, to provide upper bounds on β. Applying the same procedure would lead to

$$\alpha = \sum_{z=k}^{\frac{n(n+1)}{2}} P_{\text{SR}_i^{\theta_{\text{ic}}}}\,[z|\,\text{IC}], \tag{5.16}$$

$$\beta \leq 1 - \alpha, \tag{5.17}$$

where k is the control limit. Proving that this is actually true requires the demonstration of the following inequality

$$\text{SR}_i^{\theta_{\text{oc}}} \leq \text{SR}_i^{\theta_{\text{ic}}}, \tag{5.18}$$

that must hold for any random sample. The demonstration is as follows.

Let $(x_{i1}, x_{i2}, \ldots, x_{in})$ be a random sample drawn from any distribution, and let θ_{ic} and θ_{oc} be two values taken in the space of the observations such that $\theta_{\text{ic}} \leq \theta_{\text{oc}}$. Remember that

$$\text{SR}_i^{\theta_{\text{ic}}} = \sum_{j=1}^{n} \text{sign}\left(x_{ij} - \theta_{\text{ic}}\right) R_{ij}^{\theta_{\text{ic}}}, \tag{5.19}$$

$$\text{SR}_i^{\theta_{\text{oc}}} = \sum_{j=1}^{n} \text{sign}\left(x_{ij} - \theta_{\text{oc}}\right) R_{ij}^{\theta_{\text{oc}}}, \tag{5.20}$$

where $R_{ij}^{\theta_{\text{ic}}}$ and $R_{ij}^{\theta_{\text{oc}}}$ represent the ranks of the absolute values of the differences $x_{ij} - \theta_{\text{ic}}$ and $x_{ij} - \theta_{\text{oc}}$ respectively. We now show that, for every term in the sums above, the following relation holds : $\text{sign}\left(x_{ij} - \theta_{\text{oc}}\right) R_{ij}^{\theta_{\text{oc}}} \leq \text{sign}\left(x_{ij} - \theta_{\text{ic}}\right) R_{ij}^{\theta_{\text{ic}}}$. Indeed, three situations can be encountered:

1. if $x_{ij} < \theta_{\text{ic}}$, then $\text{sign}\left(x_{ij} - \theta_{\text{ic}}\right) = \text{sign}\left(x_{ij} - \theta_{\text{oc}}\right) = -1$, and $R_{ij}^{\theta_{\text{ic}}} \leq R_{ij}^{\theta_{\text{oc}}}$ since $|x_{ij} - \theta_{\text{ic}}| \leq |x_{ij} - \theta_{\text{oc}}|$;
2. if $x_{ij} > \theta_{\text{oc}}$, then $\text{sign}\left(x_{ij} - \theta_{\text{ic}}\right) = \text{sign}\left(x_{ij} - \theta_{\text{oc}}\right) = 1$, and $R_{ij}^{\theta_{\text{ic}}} \geq R_{ij}^{\theta_{\text{oc}}}$ since $|x_{ij} - \theta_{\text{ic}}| \geq |x_{ij} - \theta_{\text{oc}}|$;
3. if $\theta_{\text{ic}} < x_{ij} < \theta_{\text{oc}}$, then $\text{sign}\left(x_{ij} - \theta_{\text{ic}}\right) = 1$ and $\text{sign}\left(x_{ij} - \theta_{\text{oc}}\right) = -1$, while the ranks remain positive.

Note that in the third case, the ranks of the observations can either increase or decrease. If the rank increases, then it means that the rank of one observation $x_{ij} > \theta_{\text{oc}}$ decreases. Likewise, if the rank decreases, then the rank of an observation belonging to the first case $x_{ij} < \theta_{\text{ic}}$ increases.

In any case, all the terms in the sums obey the relation

$$\text{sign}\left(x_{ij} - \theta_{\text{oc}}\right) R_{ij}^{\theta_{\text{oc}}} \leq \text{sign}\left(x_{ij} - \theta_{\text{ic}}\right) R_{ij}^{\theta_{\text{ic}}},$$

thus proving Equation 5.18.

5.2 Possible improvements of the method

The main drawback of the approach proposed in the previous section is that the bound given for β is quite loose. Therefore, the statistical guarantees that we have on the probability of type II errors are not very strong. Another consequence is that the expected cost is somewhat overestimated and will not reflect the true cost faced during the operation of the process.

One possible improvement would be to determine the exact distribution of the statistic $\text{SN}_i^{\theta_{\text{ic}}}$ when the process is out-of-control, i.e. the distribution $P\left[\text{SN}_i^{\theta_{\text{ic}}} = z \middle| \text{OC}\right]$, but we do not know whether this is even feasible. Possible solutions could use parameterized distributions, such as the normal one, to characterize the out-of-control distribution of the statistic, but the nonparametric aspect of the procedure would be lost.

Another possible improvement over our procedure could be to compute two different control limits k_{ic} and k_{oc}, one for each statistic. Then, both statistics could be computed for each sample and compared to their respective control limit:

$$\mathrm{SN}_i^{\theta_{\mathrm{ic}}} \overset{?}{\geq} k_{\mathrm{ic}},$$
$$\mathrm{SN}_i^{\theta_{\mathrm{oc}}} \overset{?}{\geq} k_{\mathrm{oc}},$$

where the first and second tests ensure independently bounds on the probabilities of type I and type II errors, respectively.

If the answer to both inequalities is yes, then there is strong evidence that the process is out-of-control. If both answers are no, then the process might be considered as in-control. The problematic situation arises when the tests disagree. If the first test indicates that the process is out-of-control, and the second one says that it is in-control, the final decision that should be taken is not clear. The problem then would be to find a single rule that combines the outputs of both tests such that their statistical guarantees are maintained.

Chapter 6

Experiments

The purpose of this chapter is to present the experiments that we performed to compare parametric and nonparametric charts in the framework of economic statistical design. We first present the methodology of the experimental procedure in Section 6.1. Some important results are then reported in Section 6.2, which are then finally discussed in Section 6.3.

6.1 Methodology and practical implementation details

In order to compare parametric and nonparametric economic statistical designs, we adopt the following framework.

1. Choose one control chart.
2. Choose one experimental configuration, i.e. a distribution type for the quality characteristic, and the values of the parameters of this distribution for both in-control and out-of-control states. The parameters values are chosen in such a way that the OC distribution has a larger mean and median than the IC distribution.
3. Determine (by a closed form equation or through phase I estimation) the mean, standard deviation and median of the chosen IC and OC distributions.
4. Create the ESD problem by using the α and β functions corresponding to the chosen control chart and the estimated/computed mean, standard deviation and median of the IC and OC distributions.
5. Solve the ESD problem, and find the optimal values (k^*, n^*, h^*).
6. Implement the chosen control chart with (k^*, n^*, h^*).
7. Simulate the process a certain number of times with the chosen experimental configuration. At the beginning of each simulation, the process starts in the IC state and is affected by a certain number of assignable causes. The implemented chart is then used to detect the occurrences of the cause in order to bring the process back in-control as quickly as possible.
8. Analyze the results of all simulated runs for this experimental configuration.

The remainder of this section is focused on the technicalities of this experimental methodology. More specifically, we discuss in detail the faced practical issues, the choice of the parameters, as well as other elements that are deemed important for a good understanding of the experiments.

6.1.1 Step 1: the control chart

The first step, which conditions most of the rest of the procedure, is to choose which control chart we want to use. In this work, we consider only control charts for the mean or for the median. Moreover, we consider only Shewhart type charts, i.e. charts that do not rely on history to take a decision but only on the current sample.

We will thus consider three types of charts, one parametric and two nonparametric charts. The famous $\overline{x}$ chart is the parametric chart that we will focus on. The SN and SR charts are the studied nonparametric charts. A description of those parametric and nonparametric charts is available in Section 3.2 and Section 3.3, respectively.

6.1.2 Step 2: the distribution of the quality characteristic

The point in using nonparametric charts is to render the whole procedure independent of the type of the distribution of the quality characteristic[1]. In this work, and in order to compare the parametric and nonparametric procedures, we thus have to consider different types of distributions. Besides the normal distribution, we have considered the Cauchy, Erlang, exponential, double exponential, and mixture of normals distributions.

The exponential and Erlang distributions have been chosen for their non-symmetric probability density functions (pdf). The other distributions have been selected because prior work showed that these distributions (especially the heavy tailed distributions) are good candidates to fault the $\overline{x}$ chart on non-normal distributions (see e.g. Bakir and Reynolds, 1979; Balakrishnan and Kocherlakota, 1986).

The thorough description of these distributions is beyond the scope of this report. However, we give in Tables A.1 and A.2, on pages 64 and 65 respectively, a brief digest of the important aspects of each distribution, namely their parameters, the probability density function (pdf), the cumulative distribution function (cdf), and, if they exist, the closed form mean, standard deviation, and median.

In our experiments, we considered several so-called experimental configurations. An experimental configuration considers a single distribution type, both for the IC and OC states, according to which the quality characteristic will be distributed. We then determine the values of the parameters of the chosen distribution for the IC and OC states so that the OC mean/median is larger than the IC one. The experimental configuration is thus defined by a distribution type and two sets of parameters that characterize, in the end, the IC and OC distributions of the quality characteristic. The complete set of experimental configurations is reported in Appendix B.1. For each distribution type, the experimental configurations are numbered to be referred to more easily.

[1]Remember that the parametric control charts depend on the distribution type.

6.1.3 Step 3: characteristics of the distribution

As mentioned earlier, some characteristics of the distribution such as the mean, the standard deviation, and the median, are required in order to implement the control charts. Sometimes, those values can be computed directly from the parameters of the distributions. It is the case for the normal, exponential, and double exponential distributions.

Some other times, it is not possible to theoretically compute these characteristics. For instance, the mean and the standard deviation of the Cauchy distribution are not defined. However, the $\overline{x}$ chart still requires some values to compute the probabilities of type I and type II errors. As a result, we decided to estimate those values through the traditional method, even if these characteristics do not exist. In that case, we estimate the median as well, in order to honestly compare the parametric and nonparametric charts. Indeed, it would have been unfair to compare a chart using estimated values with another chart using theoretical values. Similarly, the median of the Erlang distribution is not available in a simple closed form. We thus use estimates of the mean, standard deviation and median within our control charts. The estimates of the characteristics of the distribution are computed through a phase I approach using 25 samples of size 5 (see Section 4.1).

The last case concerns the mixture of normals distribution. No simple closed form formula permits to compute the value of the median from the parameters. However, the value that we use is an estimate computed with a good precision from the cumulative distribution function. The value estimated with this method is assumed to be accurate enough and the theoretical mean and standard deviation together with the estimated median are therefore used for all charts. Note that this procedure could be applied to the Erlang distribution as well. However, we decided not to do it in order to have two distributions for which parameters have to be estimated. This introduces into the results some additional variability that will be interesting to analyze. Indeed, phase I parameter estimation has been identified as crucial to the performance of control charts (Jensen et al., 2006), and developing robust methods are thus an important practical task.

The closed form formulas giving the mean, the standard deviation, and the median for the distributions are provided in Appendix A.

6.1.4 Step 4: creating the ESD problem

Creating the ESD problem is very easy once the chart is chosen. Indeed, it suffices to introduce the $\alpha(k, n, h)$ and $\beta(k, n, h)$ functions of the chosen control chart into Problem (4.16), and to give specific values to the parameters describing the cost model.

We decided to consider the problem studied by Faraz et al. (2013). They focus on a delivery chain process that has to be controlled. We re-use the same set of values that were presented in their paper. This set is reported in Table 6.1.

Note that, for optimization purposes, we added bounds on the values that k, n, and h can take. For example, the maximal value of n is 50. And the maximal value of h is 30. As for the value of the control limit k, the maximum value that we allow depends on the control chart. Adding these bounds on the parameters made the problem much easier to solve. This

$\lambda = 0.003$	$C_0 = 3,150$	$C_1 = 29,637$
$W = 250$	$Y = 10,375$	$S = 10$
$E = 0.23$	$T = 0.62$	

Table 6.1: Parameters of the delivery chain process.

is due to the optimization mechanisms of the genetic algorithms that are used to optimize the cost function.

We set the values of α_0 and β_0 to 0.1 and 0.99, respectively. These large values have been chosen to make the problem easily solvable. Indeed, due to the combinatorial nature of the problem, setting aggressive statistical constraints might prevent the optimization algorithm from finding a solution. In this first study, we decided to avoid such issues by choosing large values for α_0 and β_0. Furthermore, note that, because of the way the type II error probability is computed for the nonparametric charts, the following relation must hold: $\beta_0 \geq 1 - \alpha_0$.

6.1.5 Step 5: solving the ESD problem

Problem (4.16) is a mixed-integer nonlinear problem. Solving such problems is not easy, and we must resort to special optimization techniques. This problem is solved with genetic algorithms whose description is omitted here. For a deeper understanding of the mechanisms of genetic algorithms, see, for example, Faraz et al. (2013). We set the population size to 2,000 and the maximum number of generations to 1,000. Other stopping criteria are also utilized by the genetic algorithms to stop the optimization before the number of generations is exhausted. For instance, one of these criteria is to stop the optimization algorithm if no improvement has been observed in the objective function after a certain number of generations. Because of the additional stopping criteria, the optimizations finished quite quickly in general and most of the runs did not reach 1,000 generations.

Once the optimization terminates, the optimal values (k^*, n^*, h^*) of the parameters are collected to implement the control chart. Recall that these optimal values are most likely locally optimal, since the genetic algorithms have no guarantee to find the infimum of the function. Note that, in some rare cases, the optimization did not find any feasible solution. This will be indicated in the results. In that situation, we will use the values (k, n, h) that minimize the cost function even if the solution is not feasible, i.e. does not satisfy all constraints of the problem.

6.1.6 Step 6: control chart implementation

The locally optimal values (k^*, n^*, h^*) of the parameters can now be used, together with other information about the distribution, to implement the control chart. In the case of the $\overline{x}$ chart, no additional information is required. In the case of the nonparametric charts, we must provide the control chart with the median value of the quality characteristic. This has either been computed directly from the parameters of the distribution, or estimated through phase I estimation.

6.1.7 Step 7: performing the experiments

Once the control chart is implemented, we can simulate the process with the chosen distributions of the quality characteristic. The arrival times of the assignable causes are drawn from an exponential distribution with parameter $\lambda = 0.003$. The problem first starts in-control, and we randomly generate the time of occurrence of the first assignable cause before the first in-control sample is taken. This time indicates after how long the process will go out-of-control. Before the state switches to out-of-control, all the samples are drawn from the IC distribution corresponding to the experimental configuration that we selected. For each sample, the traditional procedure is applied: first compute the statistic, and then compare the statistic to the control limit. If the computed statistic falls outside the limits, an out-of-control signal is raised. Of course, this corresponds to a false alarm, since the process is in-control.

After a certain amount of time, the switch occurs and the process goes out-of-control. In this state, the procedure is the same, except that the sample is drawn from the chosen OC distribution. Here, the number of false negatives corresponds to the number of samples between the first sample after the process went out-of-control and the sample at which the chart gives an OC signal. After the OC signal, the process takes some time to be repaired and then goes back in-control. The procedure then starts all over again from an IC state.

A time horizon is fixed to avoid running the process for ever. This time horizon corresponds to $10,000$ time units. The number of samples is given by $\frac{10{,}000}{h^*}$. Once the time horizon is exhausted, the experimental run terminates and results about the states, false and true alarms are collected in order to evaluate how well the chart performed for that particular experimental configuration. This simulation procedure is applied repeatedly $1,000$ times in order to later average the results obtained on each single run.

6.1.8 Step 8: analyzing the experimental results

Our analysis of the experiments is mostly focused on the cost of operating the process, and on the observed type I and type II errors. Each of these three elements is computed for a given experimental run, and then averaged over all the runs corresponding to the same experimental configuration, i.e. the same IC and OC distributions.

The cost is computed as follows. First, the total times that the process is in-control and out-of-control are computed, and multiplied by the cost of operating the process when it is in-control and out-of-control, respectively. The total sampling cost is computed by multiplying the sampling cost per time unit by the time horizon. Finally, the cost of false and true alarms is added depending on how many times the chart raised them. The total cost computed in that way is finally divided by the time horizon to obtain the cost per time unit.

The frequency of occurrence of false (resp. true) alarms can be easily computed by taking the ratio of the number of false (resp. true) alarms by the number of states for which the process was in-control (resp. out-of-control).

6.2 Results summary

In this section, we take a quick peek at some results summarized graphically. The entirety of the results is available in Appendix B.

The next figures illustrate the simulation results for each distribution and each experimental configuration. The experimental configurations are referred to by a number that identifies each configuration for a given distribution. For a complete description of the experimental configurations, see Appendix B.1.

In the following figures, four performance measures are reported for each experimental configuration and for each control chart. The first measure is the average cost per time unit observed during the simulations. The second one represents the relative difference between the true observed cost and the expected cost given by the model. If the average true cost of the simulations is denoted c_t and the expected cost given by the model is c_e, then the measure reported on the graph entitled 'True vs. expected cost' is given by $\frac{c_t - c_e}{c_e}$. Finally, the last two graphs show the frequency of occurrence, observed during the simulations, of false positives (false alarms) and false negatives, respectively.

The purpose of these graphs is to compare the economic statistical design of the three studied control charts for different distributions and different experimental settings. The results illustrated on the figures allow a quick comparison between the three charts in several experimental configurations, and they show the differences that may exist between several control charts based on the distribution type and on its parameters.

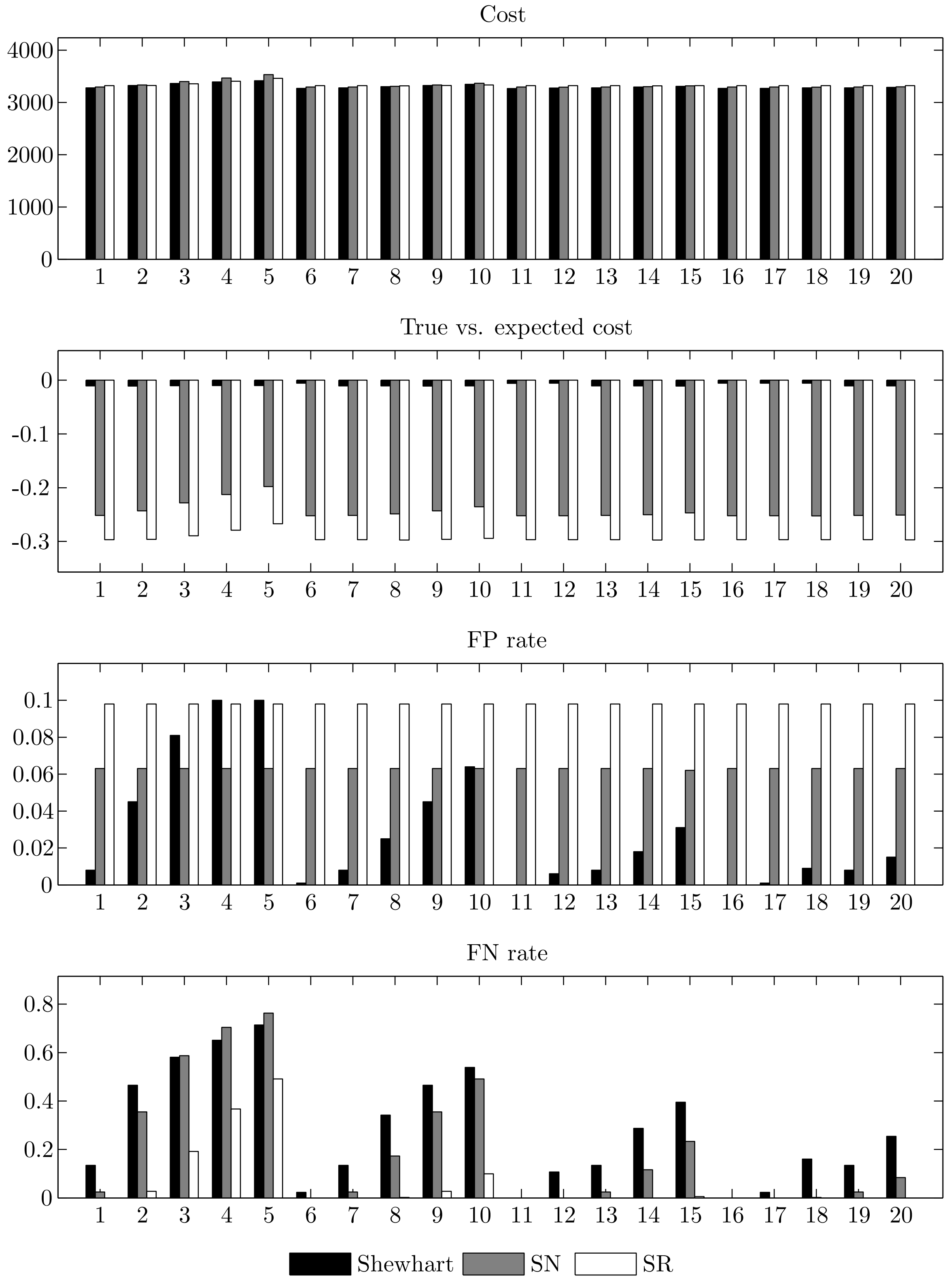


Figure 6.1: Results for normal distribution.

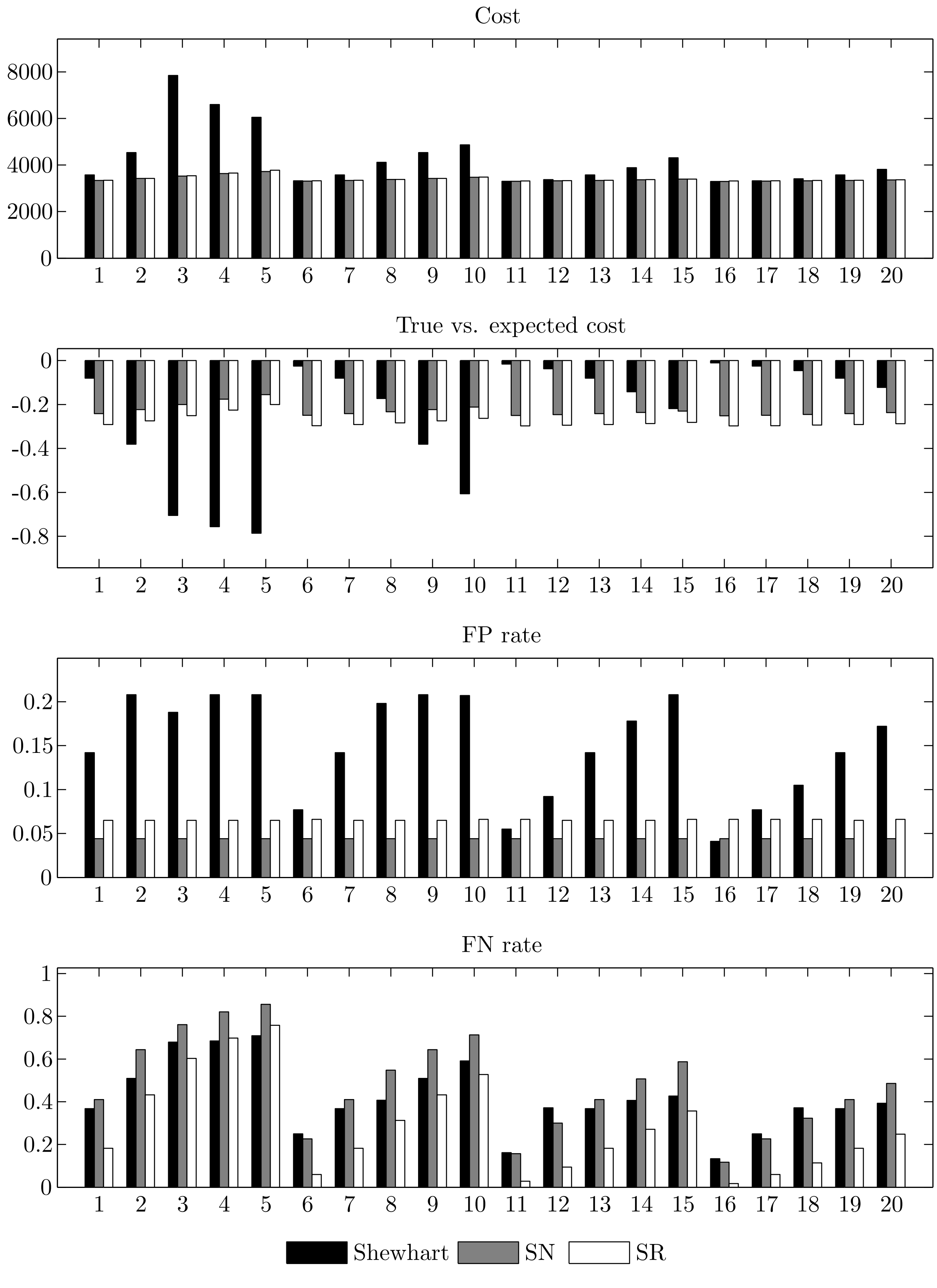


Figure 6.2: Results for Cauchy distribution.

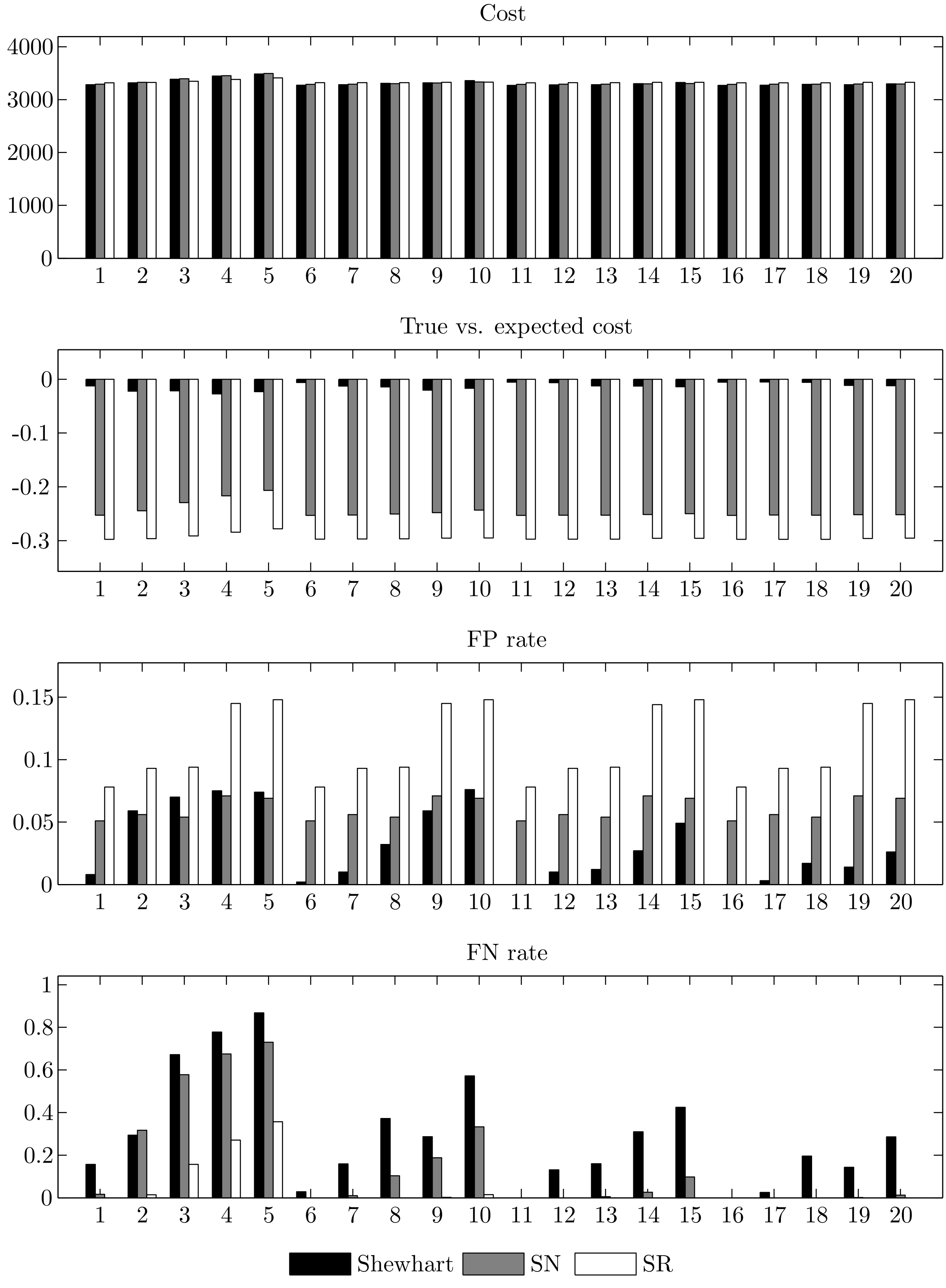


Figure 6.3: Results for Erlang distribution.

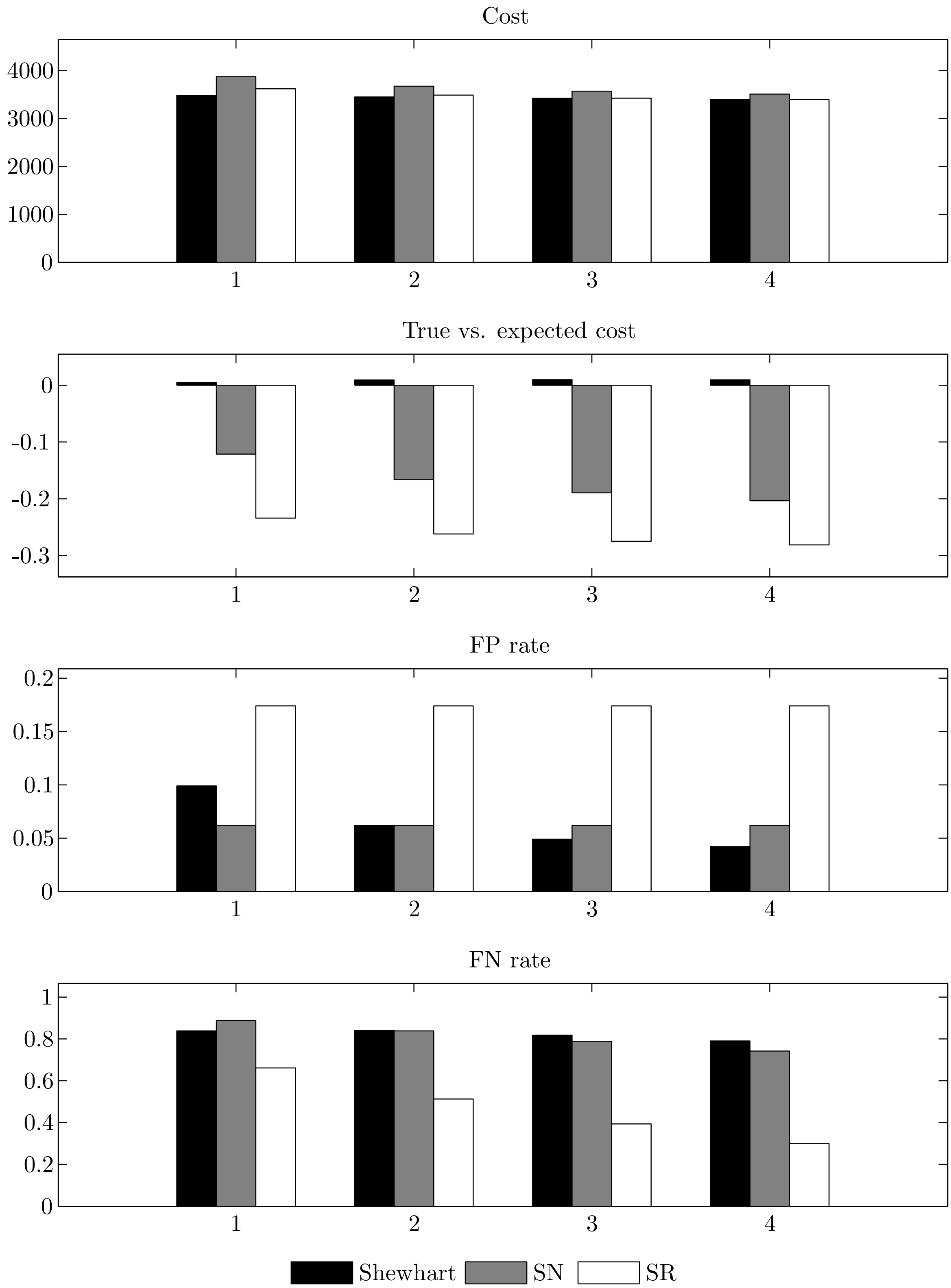


Figure 6.4: Results for exponential distribution.

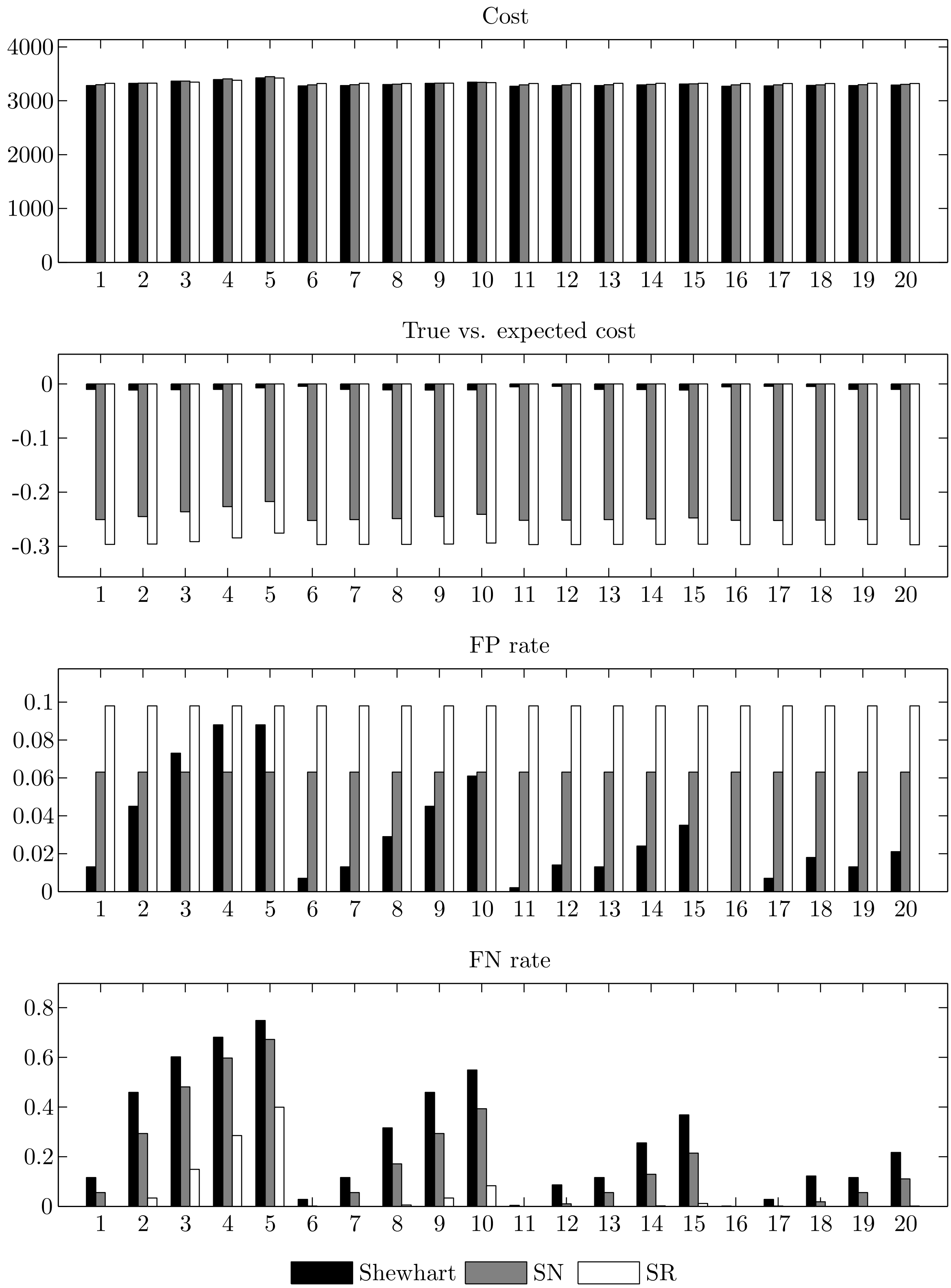


Figure 6.5: Results for double exponential distribution.

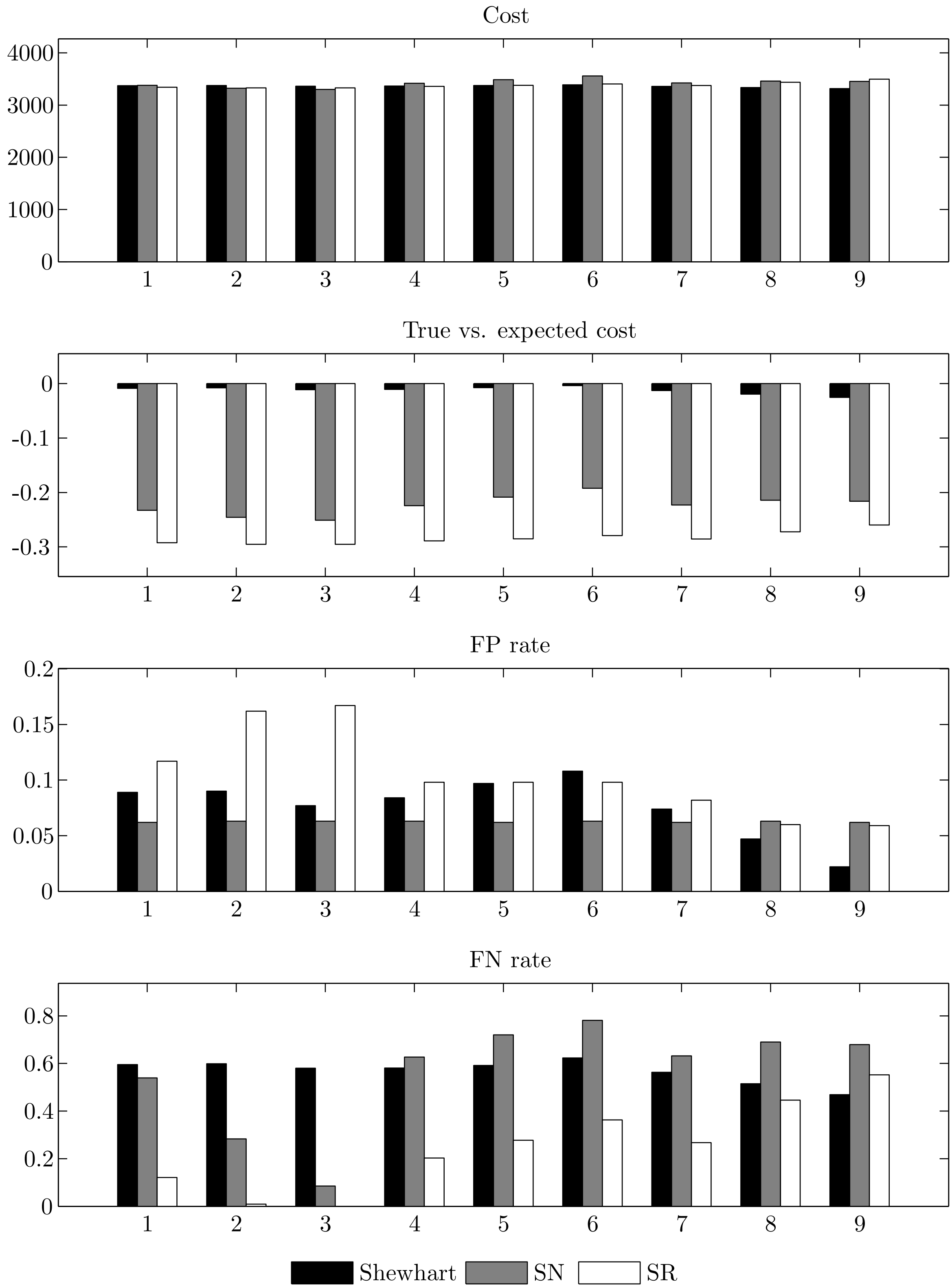


Figure 6.6: Results for mixture of normals ($\mu_{\mathrm{oc}} = 12.5, \sigma = 3$) distribution.

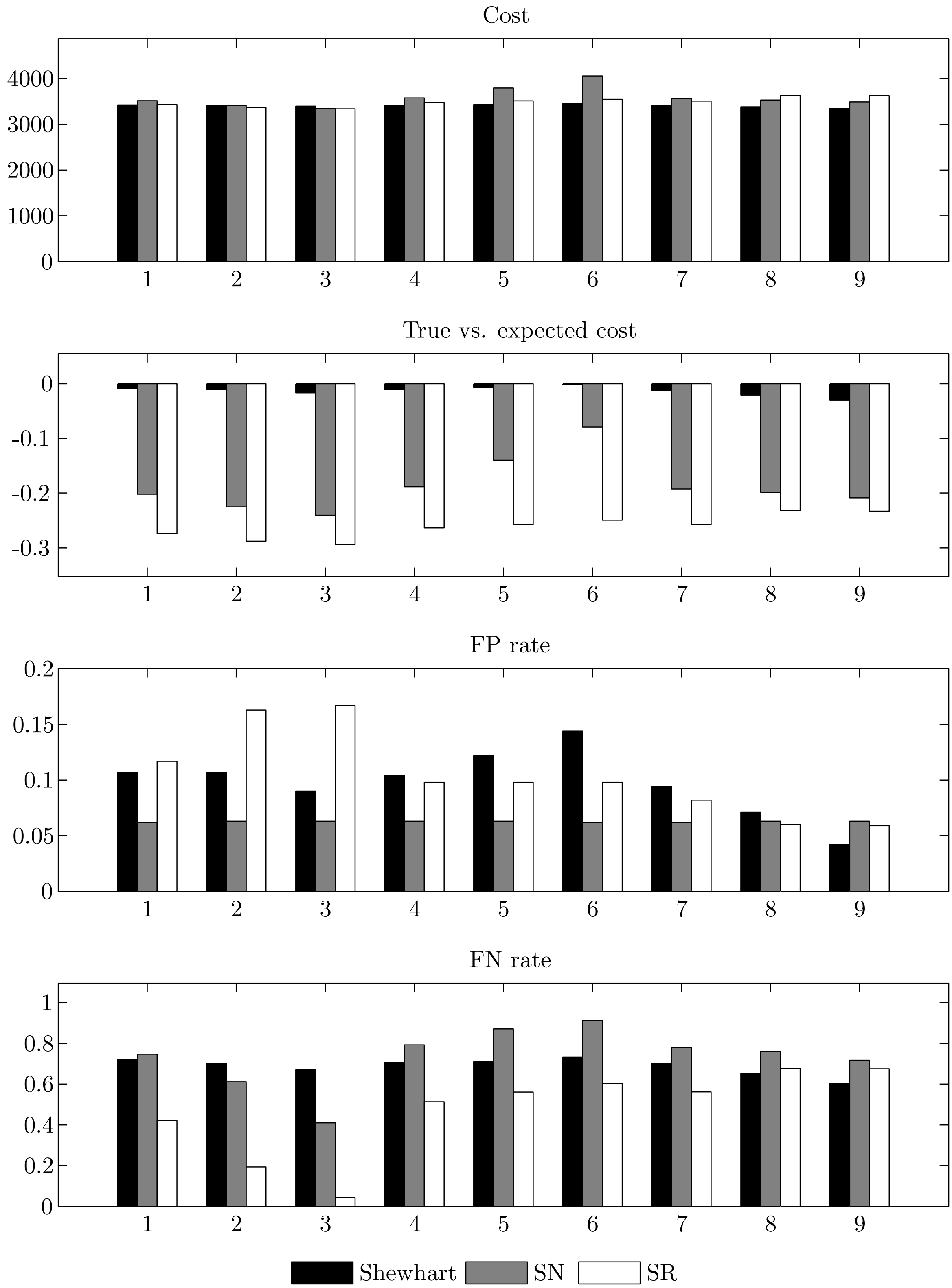


Figure 6.7: Results for mixture of normals ($\mu_{\rm oc} = 12.5, \sigma = 5$) distribution.

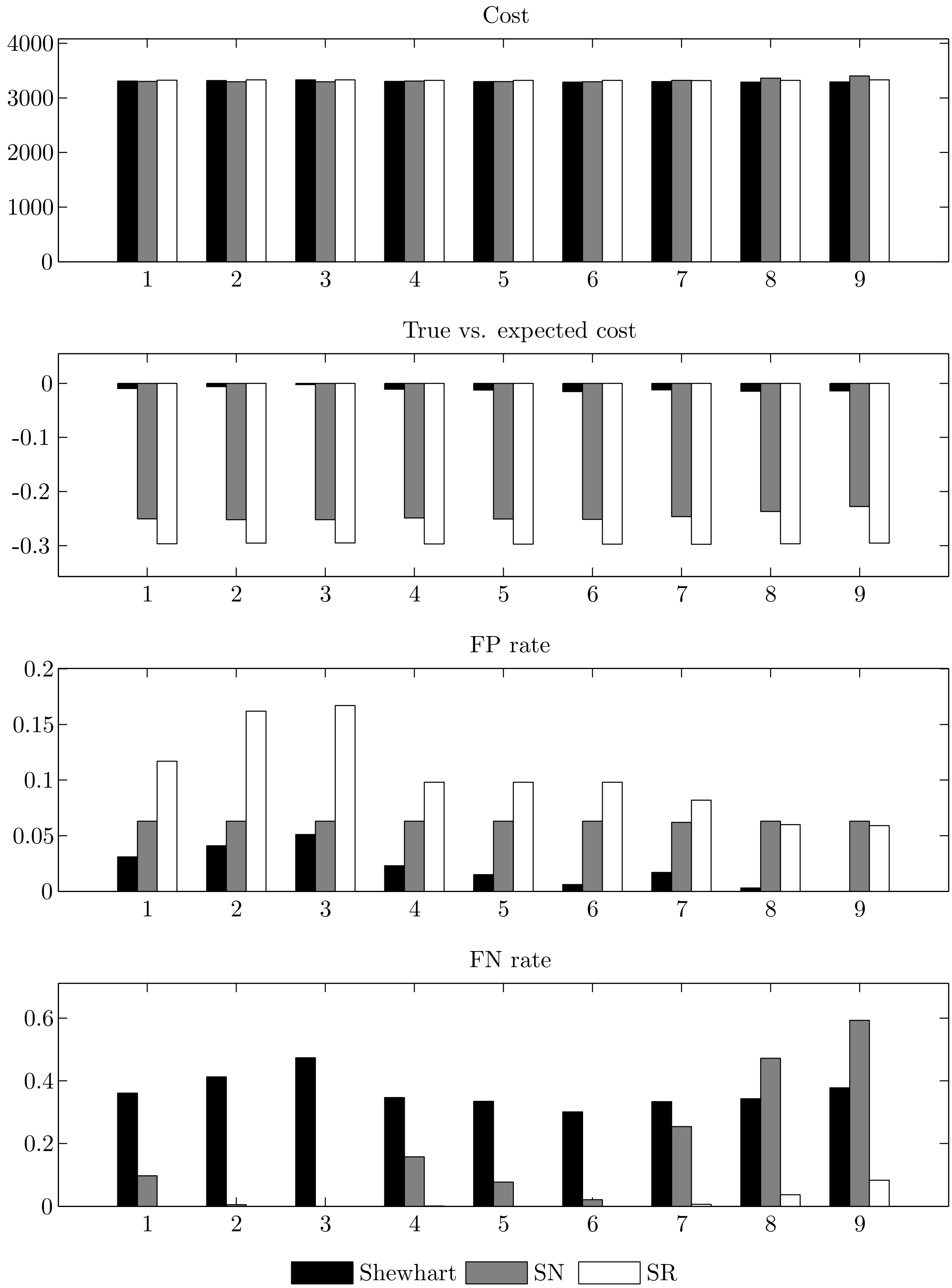


Figure 6.8: Results for mixture of normals ($\mu_{\text{oc}} = 15, \sigma = 3$) distribution.

6.3 Discussion

This section discusses some of the main elements that can be extracted from the results. The analysis carried out here is by no means exhaustive, and some interesting remarks might be missing.

6.3.1 General observations

The general observations that are made here are inferred from the results tables that can be found in Appendix B.

The first remark that can be made from these tables is that the optimal values of the design parameters k, n, and h of the nonparametric charts do not depend neither on the distribution type, nor on its parameters. On the other hand, the $\overline{x}$ chart shows a direct dependence on the distribution parameters since every single experimental configuration leads to different optimal parameters (k^*, n^*, h^*). This is a strong advantage of the nonparametric charts since the optimization does not depend on any parameter of the process. This leads to two problems in the case of $\overline{x}$ charts. First, this means that a given design of an $\overline{x}$ chart cannot be transposed to another process, unlike the nonparametric ones. Second, the efficiency of the ESD of $\overline{x}$ charts might depend on unknown parameters that need to be estimated through a phase I. Approximations in this phase will thus propagate through the entire controlling procedure affecting the control chart in an unpredictable manner. These effects can be observed for the Cauchy and the Erlang distributions for which we decided to approximate the characteristics of the distribution through a phase I. Although phase I estimation of the parameters does not influence the optimization in the nonparametric case, it does play a role in the online monitoring phase (through the estimated in-control median).

The results tables also show that there exist some discrepancies between the expected results and the results obtained through the simulations. There are essentially two causes for this.

First, the control charts usually make a certain number of assumptions on the distribution of the quality characteristic that they control. If these assumptions are not verified by the true distribution according to which the observations are drawn, then the performances of the chart will be affected. For example, the $\overline{x}$ chart assumes that the quality characteristic is distributed according to a normal distribution. Consequently, if the $\overline{x}$ chart is used to monitor a quality characteristic that is distributed according to a non-normal distribution, then the performances of the chart will be affected by the non-normality. This is not the case for the nonparametric charts since they do not assume normality. However, there might be other assumptions. For example, remember that the SR chart assumes that the distributions are symmetric. For these charts, whenever the distribution violates the assumptions made by the chart, the statistical guarantees offered by the economic statistical design do not hold anymore. This is the case for the $\overline{x}$ chart used on non-normal distributions, and for the SR chart used on non-symmetric distributions. On the other hand, the SN chart behaves as expected on all distributions, since the SN chart does not make any assumption about the distribution of the quality characteristic. The results given in the tables in Appendix B illustrate this fact.

The second source of errors arises from the phase I parameter estimation. All charts are affected by this type of errors, although nonparametric charts tend to be less affected by it because only the monitoring phase depends on the parameters of the distribution. Indeed, the parameters of the distribution do not play any role in the optimization problem for the nonparametric charts. The estimated parameters are used only during phase II, i.e. when the control chart is actually used to monitor the process. On the other hand, these parameters are central to the optimization and may play a role during phase II for the parametric control charts. This implies that an error in the estimated parameters will generally have a greater impact on the performances of the parametric charts than on the performances of the nonparametric ones. The effect of phase I estimation can be observed for the Cauchy and the Erlang distributions in the results tables. For example, the results of the SN chart for the Cauchy distribution show that there is a difference between the false positive rate computed from the experiments and the expected value of α given by model. This discrepancy between the experimental and theoretical results is due to parameters estimation.

6.3.2 Analysis of the experimental results

We move now to a more detailed study of the results. This analysis is best illustrated by the results graphs of the previous section, although the same conclusions can be drawn from the results tables.

Overall, the previous graphs show that all charts behave similarly in terms of operating costs. The cost difference between the $\overline{x}$, the SN and the SR chart is, in most cases, negligible. The only exception to this is the Cauchy distribution for which certain configurations (especially the configurations with a large scale parameter and a small difference between the in-control and out-of-control location parameter) are much more costly for the $\overline{x}$ chart than for the others. The analysis of the average cost observed during the simulations does not indicate that one chart is really better than the others. The $\overline{x}$ chart sometimes performs better than the nonparametric ones, and the opposite is true as well. Although the $\overline{x}$ chart seems to be slightly less expensive that the nonparametric charts in a greater number of situations, the difference is so small that a clear winner cannot be designated based on the analysis of the cost.

In the results graphs, the 'True vs. expected cost' graph shows the relative discrepancies between the expected cost and the average cost observed during the simulations. Overall, the difference between both costs is quite small for the $\overline{x}$ chart, but much higher for the nonparametric charts. The difference in the case of the $\overline{x}$ chart is explained by the normality assumption made by the chart which is of course not correct when the distribution is not normal. However, there is still a slight difference in the normal case between the observed and expected costs. This is due to the simulation process that differs slightly from Duncan's model. Indeed, our simulations assume for simplicity that the exponential distribution models the time interval between the last time the process has been repaired, and the next assignable cause. This introduces a small bias in the process that does not impact much the analysis. Note that, although it appears so, the observed cost is not always smaller than the expected cost. For instance, the average experimental cost of the $\overline{x}$ chart is higher than the expected cost when the distribution of the quality characteristic is an exponential. This is of course an impractical situation since the true cost can identically vary in both directions around the expected cost. On the other hand, the expected cost given by the model for the nonparametric charts is always larger than the true observed cost (if the distribution satisfies the assumptions of the charts). The expected cost is thus an upper bound for the true cost, while it is not the case for the $\overline{x}$ chart. Indeed, since the value of β used for the economic statistical design is an upper bound, the expected cost computed by the model is an upper bound on the real cost as well. The upper bound is always valid for the SN chart and it is valid for the SR chart when the distribution of the quality characteristic is symmetric. For the nonparametric charts, the 'True vs. expected cost' illustrates the difference between the true and expected costs due to the use of a bound on the type II error probability rather than the exact value. The results graphs for the nonparametric charts show that the real cost is always 20 to 30% smaller than the expected cost. Because the bound on the value of β is quite loose, so is the bound on the true cost.

Similarly to the cost study, the analysis of the frequency of false positives and false negatives is not conclusive, and no clear hierarchy can be established for the control charts. As expected, the type I error probability is indeed equal to the theoretical value given by the economic statistical model, when the distribution matches the assumptions of the chart. For the $\overline{x}$ chart, this is the case for the normal distribution. This is true for all symmetric distributions when the SR chart is used and it is always true for the SN chart (when there is no phase I estimation). As for the type II error probability, since only bounds are given for the nonparametric charts, we can only verify that the true false negative rates indeed respect the given bounds. The advantage of nonparametric charts is that the number of situations in which the expected values of α and β given by the model match the experimental results is larger than for parametric charts. Thus, if the false positive and negative rates are a main concern, using nonparametric charts might be a wise choice, since parametric charts do not guarantee, in general, false positives nor false negatives rates.

The comparison of the three charts on all experimental configurations reveals that all charts have strengths and weaknesses and that it is impossible to decide which chart is better even for a given distribution. Note that the analysis is made even more difficult by the fact that the type I and type II error probabilities are usually negatively correlated, which implies that it is generally not possible to find a chart that performs better than the other charts on both criteria.

We believe that the most important outcome of this analysis is that the nonparametric charts have the advantage over the $\overline{x}$ chart as far as the statistical guarantees are concerned. Indeed, the theoretical values given by the economic statistical model are the values that are indeed observed in practice for the nonparametric charts in most situations. This results from the weaker assumptions made by the nonparametric charts in comparison with the parametric ones. This implies that, since the cost analysis is not conclusive to determine which control chart is better, the use of nonparametric control charts might be justified when the statistical properties of the chart are crucial.

Finally, let us mention that this analysis is valid for the particular parameters of the process that we used here. If some costs, say the costs of investigating false alarms or the out-of-control operating costs, change, the results of the analysis might change dramatically as well. Consequently, it is hard to draw definitive general conclusions about the performances of the parametric versus the nonparametric charts.

Chapter 7

Conclusions and future work

In this work, we have applied the economic statistical design framework to nonparametric control charts. To this end, we have developed bounds for the type II error probability, i.e. the false negatives rate, that have been used within the economic statistical design model. We implemented the optimization problem defining this model and used it to find the design parameters of nonparametric control charts. Later, we compared the behavior of the model with parametric and nonparametric charts on different probability distributions with different values of the distribution parameters.

We then performed a brief analysis of the obtained results that emphasizes the differences between the economic statistical design of parametric and nonparametric control charts. In this study, we also gave a number of advantages and shortcomings of both approaches so that the interested reader can make the best possible decision on which control chart it is better to use for a given application.

One of the advantages of nonparametric economic statistical design is that the optimal design parameters are completely independent from the distribution type, and from the parameters of the distribution. This renders the approach very robust when compared to traditional parametric control charts. Moreover, this implies that a single efficient nonparametric design can generally be used in most situations with statistical guarantees on the type I error probability, i.e. the false positives rate.

The simulation results show that the economic statistical design of nonparametric control charts compares favorably with traditional methods. The average operating cost is roughly equal to the one obtained when parametric charts are used. The main difference is that the statistical guarantees offered by the economic statistical design are overall met with nonparametric charts, unlike parametric ones. This is a strong superiority over parametric charts since statistical guarantees are usually one of the most important criteria in the design of control charts.

Another advantage of our approach is that very little process information is required to actually use the nonparametric chart. As a matter of fact, no information at all about the process is needed during the optimization, and only the value of the in-control median is required when the process has to be controlled. This small amount of information demanded to apply the control chart is an advantage in some cases. However, when additional information is available, using this additional knowledge would probably greatly improve the efficiency of the charts. This constitutes a promising research direction to further improve the performances of economic statistical design of nonparametric control charts.

One of the main limitations of our approach is that the type II error probability is bounded, rather than computed exactly. Another possible research direction would be to determine a distribution that could be used to exactly compute the probability of having type II errors. This would clearly improve the method, by strengthening the statistical guarantees that the nonparametric charts offer. However, it is not clear whether finding such a distribution is even feasible. If the exact determination of that distribution is impossible, allowing the out-of-control distribution to be parametric might be an acceptable tradeoff to tighten a bit the bound on the type II error probability. Of course, the nonparametric nature of the control chart would then be lost.

So far, the nonparametric control charts that we used were univariate. Developing the same economic statistical design framework for multivariate nonparametric control charts is another very interesting line of future work that could find many applications in the industry.

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Appendix A

Probability distributions and their parameters

This appendix gives some useful information about the probability distributions that we used in this work. The probability distributions are shown in Table A.1 and Table A.2.

We considered the following probability distributions: the normal, the Cauchy, the double exponential, the exponential, the Erlang, and the mixture of normals distributions.

The tables contain, for each distribution,

1. the parameters of the distribution and their acceptable range;
2. the probability density function, denoted $pdf(x)$;
3. the cumulative distribution function, denoted $cdf(x)$;
4. the mean, the standard deviation, and the median, when a simple closed form exists.

Normal	parameters	the mean $\mu \in \mathbb{R}$
		the standard deviation $\sigma \in (0; +\infty)$
	pdf	$pdf(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp -\frac{(x-\mu)^2}{2\sigma^2}$
	cdf	$cdf(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
	mean	μ
	standard deviation	σ
	median	μ
Cauchy	parameters	the location $\theta \in \mathbb{R}$
		the scale $\gamma \in (0; +\infty)$
	pdf	$pdf(x) = \frac{1}{\pi\gamma\left[1+\left(\frac{x-\theta}{\gamma}\right)^2\right]}$
	cdf	$cdf(x) = \frac{1}{2} + \frac{1}{\pi} \operatorname{atan}\left(\frac{x-\theta}{\gamma}\right)$
	mean	undefined
	standard deviation	undefined
	median	θ
Erlang	parameters	the shape $k \in \mathbb{N}_0^+$
		the rate $\lambda \in (0; +\infty)$
	pdf	$pdf(x) = \frac{\lambda^k x^{k-1} \exp(-\lambda x)}{(k-1)!}$
	cdf	$cdf(x) = 1 - \sum_{n=0}^{k-1} \frac{1}{n!} \exp(-\lambda x)(\lambda x)^n$
	mean	$\frac{k}{\lambda}$
	standard deviation	$\frac{\sqrt{k}}{\lambda}$
	median	no simple closed form

Table A.1: Different distributions used in this work. Part 1.

Exponential	parameters	the rate $\lambda \in (0; +\infty)$
	pdf	$pdf(x) = \lambda \exp(-\lambda x)$
	cdf	$cdf(x) = 1 - \exp(-\lambda x)$
	mean	λ^{-1}
	standard deviation	λ^{-1}
	median	$\lambda^{-1} \ln 2$
Double exponential	parameters	the location $\mu \in \mathbb{R}$
		the scale $b \in (0; +\infty)$
	pdf	$pdf(x) = \frac{1}{2b} \exp\left(-\frac{\lvert x-\mu \rvert}{b}\right)$
	cdf	$cdf(x) = \begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right) & \text{if } x < \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right) & \text{if } x \geq \mu \end{cases}$
	mean	μ
	standard deviation	$\sqrt{2}b$
	median	μ
Mixture of normals	parameters	the mean $\mu_1 \in \mathbb{R}$
		the standard deviation $\sigma_1 \in (0; +\infty)$
		the mixture weight $p \in [0, 1]$
		the mixtures spacing $a \in [0; +\infty)$
	pdf	$pdf(x) = (1-p)\frac{1}{\sigma_1\sqrt{2\pi}} \exp -\frac{(x-\mu_1)^2}{2\sigma_1^2} + p\frac{1}{\sigma_1\sqrt{2\pi}} \exp -\frac{(x-\mu_1-a\sigma_1)^2}{2\sigma_1^2}$
	cdf	$cdf(x) = \frac{1-p}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu_1}{\sigma_1\sqrt{2}}\right)\right] + \frac{p}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu_1-a\sigma_1}{\sigma_1\sqrt{2}}\right)\right]$
	mean	$\mu_1 + ap\sigma_1$
	standard deviation	$\sigma_1\sqrt{1 + a^2 p (1-p)}$
	median	no simple closed form

Table A.2: Different distributions used in this work. Part 2.

Appendix B

Experimental results

B.1 Experimental configurations: parameters of the distributions

This section reports the different experimental configurations used to perform our experiments. For each distribution, each configuration is numbered for identification purposes.

The tables in this appendix give the values of the parameters of the distributions for the considered experimental configurations. The notation used for the parameters in these tables corresponds to the one that has been used in the Appendix A. The following table indicates which table describes the experimental configurations of a given distribution.

Distribution	Table #
Normal	B.1
Cauchy	B.2
Erlang	B.3
Exponential	B.4
Double exponential	B.5
Mixture of normals ($\mu_{\text{oc}} = 12.5$, $\sigma = 3$)	B.6
Mixture of normals ($\mu_{\text{oc}} = 12.5$, $\sigma = 5$)	B.7
Mixture of normals ($\mu_{\text{oc}} = 15$, $\sigma = 3$)	B.8

Each table also contains the values of the mean, standard deviation, and median computed for the given set of parameters, both for the in-control and out-of-control distributions. Note that, in some cases, the values are not available. This is due, in the case of the Cauchy distribution, to the nonexistence of the mean nor of the standard deviation. In the case of the Erlang distribution, there exists no simple closed form for the median, and its value is thus omitted in the corresponding table. Note that the median could still be computed from the cumulative distribution function numerically.

	Parameters				In-control			Out-of-control		
#	μ_{ic}	σ_{ic}	μ_{oc}	σ_{oc}	mean	std	median	mean	std	median
1	10	1	12.5	1	10	1	10	12.5	1	12.5
2	10	2	12.5	2	10	2	10	12.5	2	12.5
3	10	3	12.5	3	10	3	10	12.5	3	12.5
4	10	4	12.5	4	10	4	10	12.5	4	12.5
5	10	5	12.5	5	10	5	10	12.5	5	12.5
6	10	1	15	1	10	1	10	15	1	15
7	10	2	15	2	10	2	10	15	2	15
8	10	3	15	3	10	3	10	15	3	15
9	10	4	15	4	10	4	10	15	4	15
10	10	5	15	5	10	5	10	15	5	15
11	10	1	17.5	1	10	1	10	17.5	1	17.5
12	10	2	17.5	2	10	2	10	17.5	2	17.5
13	10	3	17.5	3	10	3	10	17.5	3	17.5
14	10	4	17.5	4	10	4	10	17.5	4	17.5
15	10	5	17.5	5	10	5	10	17.5	5	17.5
16	10	1	20	1	10	1	10	20	1	20
17	10	2	20	2	10	2	10	20	2	20
18	10	3	20	3	10	3	10	20	3	20
19	10	4	20	4	10	4	10	20	4	20
20	10	5	20	5	10	5	10	20	5	20

Table B.1: Normal distribution. Description of the experimental configurations.

	Parameters				In-control			Out-of-control		
#	θ_{ic}	γ_{ic}	θ_{oc}	γ_{oc}	mean	std	median	mean	std	median
1	10	1	12.5	1	-	-	10	-	-	12.5
2	10	2	12.5	2	-	-	10	-	-	12.5
3	10	3	12.5	3	-	-	10	-	-	12.5
4	10	4	12.5	4	-	-	10	-	-	12.5
5	10	5	12.5	5	-	-	10	-	-	12.5
6	10	1	15	1	-	-	10	-	-	15
7	10	2	15	2	-	-	10	-	-	15
8	10	3	15	3	-	-	10	-	-	15
9	10	4	15	4	-	-	10	-	-	15
10	10	5	15	5	-	-	10	-	-	15
11	10	1	17.5	1	-	-	10	-	-	17.5
12	10	2	17.5	2	-	-	10	-	-	17.5
13	10	3	17.5	3	-	-	10	-	-	17.5
14	10	4	17.5	4	-	-	10	-	-	17.5
15	10	5	17.5	5	-	-	10	-	-	17.5
16	10	1	20	1	-	-	10	-	-	20
17	10	2	20	2	-	-	10	-	-	20
18	10	3	20	3	-	-	10	-	-	20
19	10	4	20	4	-	-	10	-	-	20
20	10	5	20	5	-	-	10	-	-	20

Table B.2: Cauchy distribution. Description of the experimental configurations.

#	Parameters				In-control			Out-of-control		
	k_{ic}	λ_{ic}	k_{oc}	λ_{oc}	mean	std	median	mean	std	median
1	100	10	156	12.49	10	1	-	12.49	1	-
2	25	2.50	39	3.12	10	2	-	12.49	2	-
3	11	1.11	17	1.37	9.95	3	-	12.37	3	-
4	6	0.61	9	0.75	9.80	4	-	12	4	-
5	4	0.40	6	0.49	10	5	-	12.25	5	-
6	100	10	225	15	10	1	-	15	1	-
7	25	2.50	56	3.74	10	2	-	14.97	2	-
8	11	1.11	25	1.67	9.95	3	-	15	3	-
9	6	0.61	14	0.94	9.80	4	-	14.97	4	-
10	4	0.40	9	0.60	10	5	-	15	5	-
11	100	10	306	17.49	10	1	-	17.49	1	-
12	25	2.50	76	4.36	10	2	-	17.44	2	-
13	11	1.11	34	1.94	9.95	3	-	17.49	3	-
14	6	0.61	19	1.09	9.80	4	-	17.44	4	-
15	4	0.40	12	0.69	10	5	-	17.32	5	-
16	100	10	400	20	10	1	-	20	1	-
17	25	2.50	100	5	10	2	-	20	2	-
18	11	1.11	44	2.21	9.95	3	-	19.90	3	-
19	6	0.61	25	1.25	9.80	4	-	20	4	-
20	4	0.40	16	0.80	10	5	-	20	5	-

Table B.3: Erlang distribution. Description of the experimental configurations.

#	Parameters		In-control			Out-of-control		
	λ_{ic}	λ_{oc}	mean	std	median	mean	std	median
1	1/10	1/12.5	10	10	$10 * \ln 2$	12.5	12.5	$12.5 * \ln 2$
2	1/10	1/15	10	10	$10 * \ln 2$	15	15	$15 * \ln 2$
3	1/10	1/17.5	10	10	$10 * \ln 2$	17.5	17.5	$17.5 * \ln 2$
4	1/10	1/20	10	10	$10 * \ln 2$	20	20	$20 * \ln 2$

Table B.4: Exponential distribution. Description of the experimental configurations.

#	Parameters				In-control			Out-of-control		
	μ_{ic}	b_{ic}	μ_{oc}	b_{oc}	mean	std	median	mean	std	median
1	10	$1/\sqrt{2}$	12.5	$1/\sqrt{2}$	10	1	10	12.5	1	12.5
2	10	$2/\sqrt{2}$	12.5	$2/\sqrt{2}$	10	2	10	12.5	2	12.5
3	10	$3/\sqrt{2}$	12.5	$3/\sqrt{2}$	10	3	10	12.5	3	12.5
4	10	$4/\sqrt{2}$	12.5	$4/\sqrt{2}$	10	4	10	12.5	4	12.5
5	10	$5/\sqrt{2}$	12.5	$5/\sqrt{2}$	10	5	10	12.5	5	12.5
6	10	$1/\sqrt{2}$	15	$1/\sqrt{2}$	10	1	10	15	1	15
7	10	$2/\sqrt{2}$	15	$2/\sqrt{2}$	10	2	10	15	2	15
8	10	$3/\sqrt{2}$	15	$3/\sqrt{2}$	10	3	10	15	3	15
9	10	$4/\sqrt{2}$	15	$4/\sqrt{2}$	10	4	10	15	4	15
10	10	$5/\sqrt{2}$	15	$5/\sqrt{2}$	10	5	10	15	5	15
11	10	$1/\sqrt{2}$	17.5	$1/\sqrt{2}$	10	1	10	17.5	1	17.5
12	10	$2/\sqrt{2}$	17.5	$2/\sqrt{2}$	10	2	10	17.5	2	17.5
13	10	$3/\sqrt{2}$	17.5	$3/\sqrt{2}$	10	3	10	17.5	3	17.5
14	10	$4/\sqrt{2}$	17.5	$4/\sqrt{2}$	10	4	10	17.5	4	17.5
15	10	$5/\sqrt{2}$	17.5	$5/\sqrt{2}$	10	5	10	17.5	5	17.5
16	10	$1/\sqrt{2}$	20	$1/\sqrt{2}$	10	1	10	20	1	20
17	10	$2/\sqrt{2}$	20	$2/\sqrt{2}$	10	2	10	20	2	20
18	10	$3/\sqrt{2}$	20	$3/\sqrt{2}$	10	3	10	20	3	20
19	10	$4/\sqrt{2}$	20	$4/\sqrt{2}$	10	4	10	20	4	20
20	10	$5/\sqrt{2}$	20	$5/\sqrt{2}$	10	5	10	20	5	20

Table B.5: Double exponential distribution. Description of the experimental configurations.

#	Parameters						In-control			Out-of-control		
	$\mu_{1-\text{ic}}$	$\sigma_{1-\text{ic}}$	$\mu_{1-\text{oc}}$	$\sigma_{1-\text{oc}}$	p	a	mean	std	median	mean	std	median
1	8.87	2.27	11.37	2.27	0.25	2	10	3	9.73	12.50	3	12.23
2	8.50	1.50	11	1.50	0.25	4	10	3	9.15	12.50	3	11.65
3	8.38	1.08	10.88	1.08	0.25	6	10	3	8.85	12.50	3	11.35
4	7.88	2.12	10.38	2.12	0.50	2	10	3	10	12.50	3	12.50
5	7.32	1.34	9.82	1.34	0.50	4	10	3	10	12.50	3	12.50
6	7.15	0.95	9.65	0.95	0.50	6	10	3	10	12.50	3	12.50
7	6.60	2.27	9.10	2.27	0.75	2	10	3	10.27	12.50	3	12.77
8	5.50	1.50	8	1.50	0.75	4	10	3	10.85	12.50	3	13.35
9	5.15	1.08	7.65	1.08	0.75	6	10	3	11.15	12.50	3	13.65

Table B.6: Mixture of normals distribution ($\mu_{\text{oc}} = 12.5$, $\sigma = 3$). Description of the experimental configurations.

#	Parameters						In-control			Out-of-control		
	$\mu_{1-\text{ic}}$	$\sigma_{1-\text{ic}}$	$\mu_{1-\text{oc}}$	$\sigma_{1-\text{oc}}$	p	a	mean	std	median	mean	std	median
1	8.11	3.78	10.61	3.78	0.25	2	10	5	9.56	12.50	5	12.06
2	7.50	2.50	10	2.50	0.25	4	10	5	8.58	12.50	5	11.08
3	7.31	1.80	9.81	1.80	0.25	6	10	5	8.08	12.50	5	10.58
4	6.46	3.54	8.96	3.54	0.50	2	10	5	10	12.50	5	12.50
5	5.53	2.24	8.03	2.24	0.50	4	10	5	10	12.50	5	12.50
6	5.26	1.58	7.76	1.58	0.50	6	10	5	10	12.50	5	12.50
7	4.33	3.78	6.83	3.78	0.75	2	10	5	10.44	12.50	5	12.94
8	2.50	2.50	5	2.50	0.75	4	10	5	11.42	12.50	5	13.92
9	1.92	1.80	4.42	1.80	0.75	6	10	5	11.92	12.50	5	14.42

Table B.7: Mixture of normals distribution ($\mu_{\text{oc}} = 12.5$, $\sigma = 5$). Description of the experimental configurations.

#	Parameters						In-control			Out-of-control		
	$\mu_{1-\text{ic}}$	$\sigma_{1-\text{ic}}$	$\mu_{1-\text{oc}}$	$\sigma_{1-\text{oc}}$	p	a	mean	std	median	mean	std	median
1	8.87	2.27	13.87	2.27	0.25	2	10	3	9.73	15	3	14.73
2	8.50	1.50	13.50	1.50	0.25	4	10	3	9.15	15	3	14.15
3	8.38	1.08	13.38	1.08	0.25	6	10	3	8.85	15	3	13.85
4	7.88	2.12	12.88	2.12	0.50	2	10	3	10	15	3	15
5	7.32	1.34	12.32	1.34	0.50	4	10	3	10	15	3	15
6	7.15	0.95	12.15	0.95	0.50	6	10	3	10	15	3	15
7	6.60	2.27	11.60	2.27	0.75	2	10	3	10.27	15	3	15.27
8	5.50	1.50	10.50	1.50	0.75	4	10	3	10.85	15	3	15.85
9	5.15	1.08	10.15	1.08	0.75	6	10	3	11.15	15	3	16.15

Table B.8: Mixture of normals distribution ($\mu_{\text{oc}} = 15$, $\sigma = 3$). Description of the experimental configurations.

B.2 Detailed experimental results

This appendix reports the complete experimental results that we obtained during our experiments. Each line of the experimental results tables corresponds to an experimental configuration detailed in Appendix A.

The experimental procedure is split in two parts: the optimization part to find the optimal parameters k, n, and h; and the simulation part where the control chart is indeed implemented and used on a simulated process.

The results are presented in three different tables for each distribution type. The first table gives the optimal values k^*, n^*, and h^*, found by the optimization algorithm to minimize the expected cost of the model. The second table reports the expected values of α and β corresponding to the optimal values (k^*, n^*, h^*) and compares those values to the real false positives and false negatives rates observed during the experiments. The third table shows the discrepancies between the expected cost predicted by the model for the current value of the parameters and the truly experienced cost.

Table B.9 summarizes all the experimental results tables appearing in this appendix grouped by distribution and result type.

	Optimization results		Simulation results FP and FN rate		Simulation results Exp. vs. true cost	
	Table #	Page #	Table #	Page #	Table #	Page #
Normal	B.10	73	B.11	74	B.12	75
Cauchy	B.13	76	B.14	77	B.15	78
Erlang	B.16	79	B.17	80	B.18	81
Exponential	B.19	82	B.20	83	B.21	84
Double exponential	B.22	85	B.23	86	B.24	87
Mixture of normals ($\mu_{\text{oc}} = 12.5$, $\sigma = 3$)	B.25	88	B.26	89	B.27	90
Mixture of normals ($\mu_{\text{oc}} = 12.5$, $\sigma = 5$)	B.28	91	B.29	92	B.30	93
Mixture of normals ($\mu_{\text{oc}} = 15$, $\sigma = 3$)	B.31	94	B.32	95	B.33	96

Table B.9: List of all tables containing experimental results by distribution type.

#	Param.		$\overline{x}$				SN				SR			
	μ_{oc}	σ_{oc}	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*
1	12.5	1	○	11.72	2	0.48	○	3	4	0.92	○	20	8	1.84
2	12.5	2	○	12.4	2	0.46	○	3	4	0.92	○	20	8	1.84
3	12.5	3	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
4	12.5	4	○	13.62	2	0.46	○	3	4	0.92	○	20	8	1.84
5	12.5	5	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
6	15	1	○	13	1	0.5	○	3	4	0.92	○	20	8	1.84
7	15	2	○	13.43	2	0.48	○	3	4	0.92	○	20	8	1.84
8	15	3	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
9	15	4	○	14.79	2	0.46	○	3	4	0.92	○	20	8	1.84
10	15	5	○	15.37	2	0.46	○	3	4	0.92	○	20	8	1.84
11	17.5	1	○	14.09	1	0.5	○	3	4	0.92	○	20	8	1.84
12	17.5	2	○	15.01	1	0.49	○	3	4	0.92	○	20	8	1.84
13	17.5	3	○	15.15	2	0.48	○	3	4	0.92	○	20	8	1.84
14	17.5	4	○	15.91	2	0.46	○	3	4	0.92	○	20	8	1.84
15	17.5	5	○	16.58	2	0.46	○	3	4	0.92	○	20	8	1.84
16	20	1	○	15.25	1	0.5	○	3	4	0.92	○	20	8	1.84
17	20	2	○	16	1	0.5	○	3	4	0.92	○	20	8	1.84
18	20	3	○	17.06	1	0.47	○	3	4	0.92	○	20	8	1.84
19	20	4	○	16.86	2	0.48	○	3	4	0.92	○	20	8	1.84
20	20	5	○	17.65	2	0.46	○	3	4	0.92	○	20	8	1.84

Table B.10: Normal distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column ‘Opt’ indicates whether the optimization algorithm found a feasible solution (○), or not (×).

#	Param.		$\overline{x}$				SN				SR			
	μ_{oc}	σ_{oc}	α	β	FP	FN	α	β	FP	FN	α	β	FP	FN
1	12.5	1	0.008	0.134	0.008 (0.001)	0.134 (0.043)	0.062	0.938	0.063 (0.002)	0.024 (0.028)	0.098	0.902	0.098 (0.004)	0 (0.000)
2	12.5	2	0.045	0.471	0.045 (0.001)	0.465 (0.057)	0.062	0.938	0.063 (0.002)	0.355 (0.074)	0.098	0.902	0.098 (0.004)	0.027 (0.029)
3	12.5	3	0.081	0.586	0.081 (0.002)	0.581 (0.053)	0.062	0.938	0.063 (0.002)	0.587 (0.058)	0.098	0.902	0.098 (0.004)	0.192 (0.066)
4	12.5	4	0.1	0.655	0.1 (0.002)	0.651 (0.042)	0.062	0.938	0.063 (0.002)	0.704 (0.048)	0.098	0.902	0.098 (0.004)	0.367 (0.072)
5	12.5	5	0.1	0.717	0.1 (0.002)	0.714 (0.041)	0.062	0.938	0.063 (0.002)	0.763 (0.040)	0.098	0.902	0.098 (0.004)	0.491 (0.068)
6	15	1	0.001	0.023	0.001 (0.000)	0.023 (0.019)	0.062	0.938	0.063 (0.002)	0 (0.000)	0.098	0.902	0.098 (0.004)	0 (0.000)
7	15	2	0.008	0.134	0.008 (0.001)	0.134 (0.043)	0.062	0.938	0.063 (0.002)	0.024 (0.028)	0.098	0.902	0.098 (0.004)	0 (0.000)
8	15	3	0.025	0.347	0.025 (0.001)	0.342 (0.057)	0.062	0.938	0.063 (0.002)	0.173 (0.065)	0.098	0.902	0.098 (0.004)	0.002 (0.007)
9	15	4	0.045	0.471	0.045 (0.001)	0.465 (0.057)	0.062	0.938	0.063 (0.002)	0.355 (0.074)	0.098	0.902	0.098 (0.004)	0.027 (0.029)
10	15	5	0.064	0.541	0.064 (0.002)	0.539 (0.050)	0.062	0.938	0.063 (0.002)	0.491 (0.066)	0.098	0.902	0.098 (0.004)	0.099 (0.052)
11	17.5	1	0	0	0 (0.000)	0 (0.002)	0.062	0.938	0.063 (0.002)	0 (0.000)	0.098	0.902	0.098 (0.004)	0 (0.000)
12	17.5	2	0.006	0.107	0.006 (0.001)	0.107 (0.039)	0.062	0.938	0.063 (0.002)	0 (0.003)	0.098	0.902	0.098 (0.004)	0 (0.000)
13	17.5	3	0.008	0.134	0.008 (0.001)	0.134 (0.043)	0.062	0.938	0.063 (0.002)	0.024 (0.028)	0.098	0.902	0.098 (0.004)	0 (0.000)
14	17.5	4	0.018	0.287	0.018 (0.001)	0.287 (0.053)	0.062	0.938	0.063 (0.002)	0.116 (0.053)	0.098	0.902	0.098 (0.004)	0 (0.002)
15	17.5	5	0.031	0.397	0.031 (0.001)	0.395 (0.056)	0.062	0.938	0.062 (0.002)	0.233 (0.069)	0.098	0.902	0.098 (0.004)	0.005 (0.012)
16	20	1	0	0	0 (0.000)	0 (0.000)	0.062	0.938	0.063 (0.002)	0 (0.000)	0.098	0.902	0.098 (0.004)	0 (0.000)
17	20	2	0.001	0.023	0.001 (0.000)	0.023 (0.019)	0.062	0.938	0.063 (0.002)	0 (0.000)	0.098	0.902	0.098 (0.004)	0 (0.000)
18	20	3	0.009	0.163	0.009 (0.001)	0.16 (0.045)	0.062	0.938	0.063 (0.002)	0.002 (0.008)	0.098	0.902	0.098 (0.004)	0 (0.000)
19	20	4	0.008	0.134	0.008 (0.001)	0.134 (0.043)	0.062	0.938	0.063 (0.002)	0.024 (0.028)	0.098	0.902	0.098 (0.004)	0 (0.000)
20	20	5	0.015	0.253	0.015 (0.001)	0.254 (0.051)	0.062	0.938	0.063 (0.002)	0.084 (0.047)	0.098	0.902	0.098 (0.004)	0 (0.001)

Table B.11: Normal distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

	Param.		$\overline{x}$		SN		SR	
#	μ_{oc}	σ_{oc}	Exp. cost	Cost	Exp. cost	Cost	Exp. cost	Cost
1	12.5	1	3,316.1	3,280.6 (19.9)	4,405.5	3,296.8 (21.7)	4,723.9	3,321.8 (28.6)
2	12.5	2	3,363	3,326.1 (25.8)	4,405.5	3,334.4 (30.9)	4,723.9	3,325.1 (30.5)
3	12.5	3	3,401.6	3,365.7 (29.0)	4,405.5	3,400.2 (48.6)	4,723.9	3,357.1 (37.2)
4	12.5	4	3,428.9	3,394.3 (33.3)	4,405.5	3,468.7 (64.4)	4,723.9	3,406.1 (49.6)
5	12.5	5	3,451.9	3,416.9 (41.2)	4,405.5	3,533.4 (81.8)	4,723.9	3,462.3 (66.0)
6	15	1	3,289.7	3,271 (19.1)	4,405.5	3,294.6 (21.5)	4,723.9	3,321.8 (28.6)
7	15	2	3,316.1	3,280.6 (19.9)	4,405.5	3,296.8 (21.7)	4,723.9	3,321.8 (28.6)
8	15	3	3,339.1	3,302.5 (22.8)	4,405.5	3,309.9 (25.9)	4,723.9	3,320.1 (27.8)
9	15	4	3,363	3,326.1 (25.8)	4,405.5	3,334.4 (30.9)	4,723.9	3,325.1 (30.5)
10	15	5	3,384	3,347.7 (28.4)	4,405.5	3,367.9 (39.4)	4,723.9	3,334.6 (32.9)
11	17.5	1	3,288.2	3,268.5 (18.0)	4,405.5	3,294.6 (21.5)	4,723.9	3,321.8 (28.6)
12	17.5	2	3,295.9	3,276.9 (19.1)	4,405.5	3,294.2 (21.4)	4,723.9	3,321.8 (28.6)
13	17.5	3	3,316.1	3,280.6 (19.9)	4,405.5	3,296.8 (21.7)	4,723.9	3,321.8 (28.6)
14	17.5	4	3,330.9	3,294.6 (21.1)	4,405.5	3,303.2 (23.0)	4,723.9	3,319.3 (28.1)
15	17.5	5	3,347.2	3,310.3 (23.6)	4,405.5	3,317.9 (28.3)	4,723.9	3,321 (29.8)
16	20	1	3,288.1	3,269.1 (18.1)	4,405.5	3,294.6 (21.5)	4,723.9	3,321.8 (28.6)
17	20	2	3,289.7	3,271 (19.1)	4,405.5	3,294.6 (21.5)	4,723.9	3,321.8 (28.6)
18	20	3	3,300.4	3,281.4 (18.6)	4,405.5	3,293.3 (21.4)	4,723.9	3,321.8 (28.6)
19	20	4	3,316.1	3,280.6 (19.9)	4,405.5	3,296.8 (21.7)	4,723.9	3,321.8 (28.6)
20	20	5	3,326.9	3,290.8 (21.2)	4,405.5	3,300.4 (23.8)	4,723.9	3,320.7 (29.0)

Table B.12: Normal distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.

	Param.		$\overline{x}$				SN				SR			
#	θ_{oc}	γ_{oc}	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*
1	12.5	1	○	12.09	17	3.92	○	3	4	0.92	○	20	8	1.84
2	12.5	2	○	12.62	50	11.5	○	3	4	0.92	○	20	8	1.84
3	12.5	3	×	14.48	44	28.88	○	3	4	0.92	○	20	8	1.84
4	12.5	4	×	15.24	50	18.88	○	3	4	0.92	○	20	8	1.84
5	12.5	5	×	16.54	50	13.78	○	3	4	0.92	○	20	8	1.84
6	15	1	○	14.06	4	0.92	○	3	4	0.92	○	20	8	1.84
7	15	2	○	14.18	17	3.92	○	3	4	0.92	○	20	8	1.84
8	15	3	○	14.18	43	9.9	○	3	4	0.92	○	20	8	1.84
9	15	4	○	15.24	50	11.5	○	3	4	0.92	○	20	8	1.84
10	15	5	○	16.6	49	11.29	○	3	4	0.92	○	20	8	1.84
11	17.5	1	○	15.71	3	0.69	○	3	4	0.92	○	20	8	1.84
12	17.5	2	○	16.72	6	1.38	○	3	4	0.92	○	20	8	1.84
13	17.5	3	○	16.27	17	3.92	○	3	4	0.92	○	20	8	1.84
14	17.5	4	○	16.41	31	7.18	○	3	4	0.92	○	20	8	1.84
15	17.5	5	○	16.54	50	11.5	○	3	4	0.92	○	20	8	1.84
16	20	1	○	17.78	2	0.61	○	3	4	0.92	○	20	8	1.84
17	20	2	○	18.13	4	0.92	○	3	4	0.92	○	20	8	1.84
18	20	3	○	18.82	8	1.86	○	3	4	0.92	○	20	8	1.84
19	20	4	○	18.36	17	3.92	○	3	4	0.92	○	20	8	1.84
20	20	5	○	18.37	28	6.51	○	3	4	0.92	○	20	8	1.84

Table B.13: Cauchy distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column ‘Opt’ indicates whether the optimization algorithm found a feasible solution (○), or not (×).

#	Param.		$\overline{x}$				SN				SR			
	θ_{oc}	γ_{oc}	α	β	FP	FN	α	β	FP	FN	α	β	FP	FN
1	12.5	1	0.1	0.395	0.142 (0.007)	0.368 (0.071)	0.062	0.938	0.044 (0.002)	0.41 (0.070)	0.098	0.902	0.065 (0.003)	0.182 (0.065)
2	12.5	2	0.1	0.793	0.208 (0.014)	0.509 (0.067)	0.062	0.938	0.044 (0.002)	0.644 (0.054)	0.098	0.902	0.065 (0.003)	0.432 (0.070)
3	12.5	3	0.08	0.989	0.188 (0.024)	0.679 (0.057)	0.062	0.938	0.044 (0.002)	0.761 (0.039)	0.098	0.902	0.065 (0.003)	0.603 (0.058)
4	12.5	4	0.1	0.994	0.208 (0.019)	0.685 (0.054)	0.062	0.938	0.044 (0.002)	0.821 (0.031)	0.098	0.902	0.065 (0.003)	0.698 (0.050)
5	12.5	5	0.1	0.998	0.208 (0.016)	0.709 (0.053)	0.062	0.938	0.044 (0.002)	0.856 (0.025)	0.098	0.902	0.065 (0.003)	0.758 (0.040)
6	15	1	0.1	0.297	0.077 (0.003)	0.25 (0.068)	0.062	0.938	0.044 (0.002)	0.226 (0.069)	0.098	0.902	0.066 (0.003)	0.059 (0.041)
7	15	2	0.1	0.395	0.142 (0.007)	0.368 (0.071)	0.062	0.938	0.044 (0.002)	0.41 (0.070)	0.098	0.902	0.065 (0.003)	0.182 (0.065)
8	15	3	0.1	0.47	0.198 (0.013)	0.407 (0.075)	0.062	0.938	0.044 (0.002)	0.548 (0.063)	0.098	0.902	0.065 (0.003)	0.313 (0.068)
9	15	4	0.1	0.793	0.208 (0.014)	0.509 (0.067)	0.062	0.938	0.044 (0.002)	0.644 (0.054)	0.098	0.902	0.065 (0.003)	0.432 (0.070)
10	15	5	0.1	0.934	0.207 (0.014)	0.591 (0.061)	0.062	0.938	0.044 (0.002)	0.713 (0.045)	0.098	0.902	0.066 (0.003)	0.527 (0.065)
11	17.5	1	0.056	0.153	0.055 (0.002)	0.162 (0.063)	0.062	0.938	0.044 (0.002)	0.157 (0.058)	0.098	0.902	0.066 (0.003)	0.028 (0.030)
12	17.5	2	0.1	0.444	0.092 (0.003)	0.372 (0.074)	0.062	0.938	0.044 (0.002)	0.3 (0.075)	0.098	0.902	0.065 (0.003)	0.094 (0.049)
13	17.5	3	0.1	0.395	0.142 (0.007)	0.368 (0.071)	0.062	0.938	0.044 (0.002)	0.41 (0.070)	0.098	0.902	0.065 (0.003)	0.182 (0.065)
14	17.5	4	0.1	0.474	0.178 (0.010)	0.406 (0.071)	0.062	0.938	0.044 (0.002)	0.507 (0.066)	0.098	0.902	0.065 (0.003)	0.271 (0.071)
15	17.5	5	0.1	0.555	0.208 (0.014)	0.427 (0.070)	0.062	0.938	0.044 (0.002)	0.587 (0.059)	0.098	0.902	0.066 (0.003)	0.357 (0.071)
16	20	1	0.037	0.142	0.041 (0.002)	0.133 (0.042)	0.062	0.938	0.044 (0.002)	0.117 (0.056)	0.098	0.902	0.066 (0.003)	0.017 (0.023)
17	20	2	0.1	0.297	0.077 (0.003)	0.25 (0.068)	0.062	0.938	0.044 (0.002)	0.226 (0.069)	0.098	0.902	0.066 (0.003)	0.059 (0.041)
18	20	3	0.1	0.435	0.105 (0.004)	0.372 (0.072)	0.062	0.938	0.044 (0.002)	0.323 (0.072)	0.098	0.902	0.066 (0.003)	0.114 (0.053)
19	20	4	0.1	0.395	0.142 (0.007)	0.368 (0.071)	0.062	0.938	0.044 (0.002)	0.41 (0.070)	0.098	0.902	0.065 (0.003)	0.182 (0.065)
20	20	5	0.1	0.433	0.172 (0.010)	0.393 (0.071)	0.062	0.938	0.044 (0.002)	0.486 (0.069)	0.098	0.902	0.066 (0.003)	0.248 (0.069)

Table B.14: Cauchy distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

#	Param.		$\overline{x}$		SN		SR	
	θ_{oc}	γ_{oc}	Exp. cost	Cost	Exp. cost	Cost	Exp. cost	Cost
1	12.5	1	3,889.3	3,578.9 (91.8)	4,405.5	3,342.3 (34.9)	4,723.9	3,349 (38.3)
2	12.5	2	7,324.3	4,538.4 (332.6)	4,405.5	3,422.9 (55.5)	4,723.9	3,429 (59.9)
3	12.5	3	26,636.7	7,851 (1050.5)	4,405.5	3,523.9 (78.6)	4,723.9	3,541.5 (88.2)
4	12.5	4	27,078.8	6,600.8 (818.3)	4,405.5	3,633.3 (105.1)	4,723.9	3,657.6 (116.0)
5	12.5	5	28,296.4	6,052.8 (698.9)	4,405.5	3,723.5 (128.1)	4,723.9	3,778.7 (145.5)
6	15	1	3,407.3	3,322 (28.2)	4,405.5	3,309 (27.3)	4,723.9	3,324.5 (32.1)
7	15	2	3,889.3	3,578.9 (91.8)	4,405.5	3,342.3 (34.9)	4,723.9	3,349 (38.3)
8	15	3	4,978.7	4,120.9 (235.2)	4,405.5	3,379.1 (44.7)	4,723.9	3,383.3 (45.9)
9	15	4	7,324.3	4,538.4 (332.6)	4,405.5	3,422.9 (55.5)	4,723.9	3,429 (59.9)
10	15	5	12,381	4,869.6 (404.3)	4,405.5	3,472.9 (66.7)	4,723.9	3,481.4 (69.5)
11	17.5	1	3,356.4	3,302.4 (22.1)	4,405.5	3,303.6 (25.6)	4,723.9	3,318.6 (29.5)
12	17.5	2	3,504.2	3,374.6 (41.5)	4,405.5	3,322.9 (28.8)	4,723.9	3,332.6 (32.4)
13	17.5	3	3,889.3	3,578.9 (91.8)	4,405.5	3,342.3 (34.9)	4,723.9	3,349 (38.3)
14	17.5	4	4,528.8	3,887.2 (164.0)	4,405.5	3,367.1 (40.5)	4,723.9	3,371.5 (45.1)
15	17.5	5	5,518	4,312.3 (271.9)	4,405.5	3,393.7 (47.3)	4,723.9	3,397.1 (50.2)
16	20	1	3,330.1	3,295.4 (19.7)	4,405.5	3,299.1 (23.8)	4,723.9	3,320 (30.0)
17	20	2	3,407.3	3,322 (28.2)	4,405.5	3,309 (27.3)	4,723.9	3,324.5 (32.1)
18	20	3	3,577.1	3,413 (51.1)	4,405.5	3,325.9 (30.5)	4,723.9	3,337.2 (33.7)
19	20	4	3,889.3	3,578.9 (91.8)	4,405.5	3,342.3 (34.9)	4,723.9	3,349 (38.3)
20	20	5	4,346.7	3,816.7 (150.5)	4,405.5	3,361 (39.5)	4,723.9	3,365.1 (42.3)

Table B.15: Cauchy distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.

	Param.		$\overline{x}$				SN				SR			
#	k_{oc}	λ_{oc}	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*
1	156	12.49	○	11.78	2	0.48	○	3	4	0.92	○	20	8	1.84
2	39	3.12	○	11.87	3	0.69	○	3	4	0.92	○	20	8	1.84
3	17	1.37	○	13.24	2	0.47	○	3	4	0.92	○	20	8	1.84
4	9	0.75	○	14.1	2	0.49	○	3	4	0.92	○	20	8	1.84
5	6	0.49	○	17.9	1	0.23	○	3	4	0.92	○	20	8	1.84
6	225	15	○	13.13	1	0.5	○	3	4	0.92	○	20	8	1.84
7	56	3.74	○	13.57	2	0.48	○	3	4	0.92	○	20	8	1.84
8	25	1.67	○	14.23	2	0.47	○	3	4	0.92	○	20	8	1.84
9	14	0.94	○	13.63	3	0.69	○	3	4	0.92	○	20	8	1.84
10	9	0.6	○	15.42	2	0.46	○	3	4	0.92	○	20	8	1.84
11	306	17.49	○	14.26	1	0.5	○	3	4	0.92	○	20	8	1.84
12	76	4.36	○	15.22	1	0.48	○	3	4	0.92	○	20	8	1.84
13	34	1.94	○	15.37	2	0.48	○	3	4	0.92	○	20	8	1.84
14	19	1.09	○	15.95	2	0.46	○	3	4	0.92	○	20	8	1.84
15	12	0.69	○	16.47	2	0.48	○	3	4	0.92	○	20	8	1.84
16	400	20	○	15.48	1	0.5	○	3	4	0.92	○	20	8	1.84
17	100	5	○	16.31	1	0.5	○	3	4	0.92	○	20	8	1.84
18	44	2.21	○	17.33	1	0.48	○	3	4	0.92	○	20	8	1.84
19	25	1.25	○	17.04	2	0.49	○	3	4	0.92	○	20	8	1.84
20	16	0.8	○	17.91	2	0.47	○	3	4	0.92	○	20	8	1.84

Table B.16: Erlang distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column 'Opt' indicates whether the optimization algorithm found a feasible solution (○), or not (×).

#	Param.		$\overline{x}$				SN				SR			
	k_{oc}	λ_{oc}	α	β	FP	FN	α	β	FP	FN	α	β	FP	FN
1	156	12.49	0.014	0.206	0.008 (0.001)	0.157 (0.046)	0.062	0.938	0.051 (0.002)	0.016 (0.023)	0.098	0.902	0.078 (0.004)	0 (0.000)
2	39	3.12	0.091	0.374	0.059 (0.002)	0.294 (0.072)	0.062	0.938	0.056 (0.002)	0.317 (0.073)	0.098	0.902	0.093 (0.004)	0.014 (0.021)
3	17	1.37	0.1	0.732	0.07 (0.002)	0.672 (0.044)	0.062	0.938	0.054 (0.002)	0.578 (0.059)	0.098	0.902	0.094 (0.004)	0.157 (0.060)
4	9	0.75	0.1	0.83	0.075 (0.002)	0.778 (0.033)	0.062	0.938	0.071 (0.002)	0.675 (0.053)	0.098	0.902	0.145 (0.005)	0.271 (0.070)
5	6	0.49	0.089	0.905	0.074 (0.001)	0.868 (0.021)	0.062	0.938	0.069 (0.003)	0.73 (0.046)	0.098	0.902	0.148 (0.005)	0.357 (0.071)
6	225	15	0.003	0.041	0.002 (0.000)	0.028 (0.022)	0.062	0.938	0.051 (0.002)	0 (0.000)	0.098	0.902	0.078 (0.004)	0 (0.000)
7	56	3.74	0.015	0.205	0.01 (0.001)	0.159 (0.047)	0.062	0.938	0.056 (0.002)	0.01 (0.018)	0.098	0.902	0.093 (0.004)	0 (0.000)
8	25	1.67	0.045	0.427	0.032 (0.001)	0.372 (0.057)	0.062	0.938	0.054 (0.002)	0.103 (0.056)	0.098	0.902	0.094 (0.004)	0 (0.002)
9	14	0.94	0.083	0.348	0.059 (0.002)	0.287 (0.071)	0.062	0.938	0.071 (0.003)	0.188 (0.063)	0.098	0.902	0.145 (0.005)	0.002 (0.007)
10	9	0.6	0.1	0.617	0.076 (0.002)	0.572 (0.049)	0.062	0.938	0.069 (0.002)	0.333 (0.072)	0.098	0.902	0.148 (0.005)	0.015 (0.022)
11	306	17.49	0	0.001	0 (0.000)	0 (0.002)	0.062	0.938	0.051 (0.002)	0 (0.000)	0.098	0.902	0.078 (0.004)	0 (0.000)
12	76	4.36	0.011	0.159	0.01 (0.001)	0.131 (0.043)	0.062	0.938	0.056 (0.002)	0 (0.001)	0.098	0.902	0.093 (0.004)	0 (0.000)
13	34	1.94	0.015	0.201	0.012 (0.001)	0.16 (0.044)	0.062	0.938	0.054 (0.002)	0.005 (0.012)	0.098	0.902	0.094 (0.004)	0 (0.000)
14	19	1.09	0.031	0.354	0.027 (0.001)	0.31 (0.053)	0.062	0.938	0.071 (0.003)	0.026 (0.028)	0.098	0.902	0.144 (0.005)	0 (0.000)
15	12	0.69	0.061	0.468	0.049 (0.001)	0.425 (0.055)	0.062	0.938	0.069 (0.002)	0.098 (0.050)	0.098	0.902	0.148 (0.005)	0 (0.002)
16	400	20	0	0	0 (0.000)	0 (0.000)	0.062	0.938	0.051 (0.002)	0 (0.000)	0.098	0.902	0.078 (0.004)	0 (0.000)
17	100	5	0.003	0.039	0.003 (0.000)	0.025 (0.021)	0.062	0.938	0.056 (0.002)	0 (0.000)	0.098	0.902	0.093 (0.004)	0 (0.000)
18	44	2.21	0.017	0.227	0.017 (0.001)	0.196 (0.050)	0.062	0.938	0.054 (0.002)	0 (0.001)	0.098	0.902	0.094 (0.004)	0 (0.000)
19	25	1.25	0.013	0.181	0.014 (0.001)	0.143 (0.044)	0.062	0.938	0.071 (0.002)	0.001 (0.007)	0.098	0.902	0.145 (0.005)	0 (0.000)
20	16	0.8	0.028	0.329	0.026 (0.001)	0.286 (0.054)	0.062	0.938	0.069 (0.002)	0.012 (0.019)	0.098	0.902	0.148 (0.005)	0 (0.000)

Table B.17: Erlang distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

	Param.		$\overline{x}$		SN		SR	
#	k_{oc}	λ_{oc}	Exp. cost	Cost	Exp. cost	Cost	Exp. cost	Cost
1	156	12.49	3,323.4	3,282.4 (18.8)	4,405.5	3,292.4 (22.3)	4,723.9	3,318.6 (28.2)
2	39	3.12	3,391.4	3,316.2 (24.6)	4,405.5	3,327.6 (30.2)	4,723.9	3,324.4 (29.5)
3	17	1.37	3,459.5	3,385 (36.6)	4,405.5	3,394.9 (47.3)	4,723.9	3,348.2 (36.5)
4	9	0.75	3,543.2	3,446.4 (51.6)	4,405.5	3,451.1 (59.8)	4,723.9	3,381.3 (42.5)
5	6	0.49	3,568	3,485.8 (43.1)	4,405.5	3,494.4 (75.8)	4,723.9	3,409.8 (48.9)
6	225	15	3,291.2	3,271.5 (18.4)	4,405.5	3,290.3 (21.2)	4,723.9	3,319.8 (29.6)
7	56	3.74	3,323.6	3,281.6 (19.5)	4,405.5	3,293.4 (20.9)	4,723.9	3,321 (28.6)
8	25	1.67	3,357.6	3,309.1 (23.5)	4,405.5	3,301.9 (24.5)	4,723.9	3,322.3 (29.4)
9	14	0.94	3,385	3,316.5 (24.3)	4,405.5	3,313.3 (24.9)	4,723.9	3,328.6 (28.8)
10	9	0.6	3,418.8	3,360.7 (29.8)	4,405.5	3,334.1 (30.9)	4,723.9	3,329.8 (29.0)
11	306	17.49	3,288.2	3,270 (19.4)	4,405.5	3,290.2 (21.7)	4,723.9	3,319.3 (29.3)
12	76	4.36	3,301	3,279.6 (19.0)	4,405.5	3,292.5 (21.7)	4,723.9	3,319.4 (28.1)
13	34	1.94	3,323.3	3,282.2 (19.1)	4,405.5	3,291.9 (21.2)	4,723.9	3,319.6 (28.9)
14	19	1.09	3,343	3,300.8 (20.5)	4,405.5	3,297.6 (22.5)	4,723.9	3,327.1 (27.6)
15	12	0.69	3,371.2	3,323.8 (25.7)	4,405.5	3,303.8 (23.0)	4,723.9	3,326.8 (28.6)
16	400	20	3,288.1	3,270.3 (18.3)	4,405.5	3,290.2 (21.7)	4,723.9	3,318.5 (28.3)
17	100	5	3,291.1	3,273.7 (18.3)	4,405.5	3,293 (21.2)	4,723.9	3,318.5 (28.7)
18	44	2.21	3,308	3,289 (19.8)	4,405.5	3,291.9 (21.8)	4,723.9	3,318 (27.2)
19	25	1.25	3,321.7	3,283.3 (19.5)	4,405.5	3,296.2 (22.1)	4,723.9	3,326.2 (27.7)
20	16	0.8	3,339.4	3,299.1 (21.3)	4,405.5	3,296.4 (22.3)	4,723.9	3,328.4 (30.2)

Table B.18: Erlang distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.

#	Param. λ_{oc}	$\overline{x}$ Opt	k^*	n^*	h^*	SN Opt	k^*	n^*	h^*	SR Opt	k^*	n^*	h^*
1	1/12.5	○	23.13	1	0.31	○	3	4	0.92	○	20	8	1.84
2	1/15	○	27.89	1	0.23	○	3	4	0.92	○	20	8	1.84
3	1/17.5	○	30.1	1	0.23	○	3	4	0.92	○	20	8	1.84
4	1/20	○	31.58	1	0.24	○	3	4	0.92	○	20	8	1.84

Table B.19: Exponential distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column 'Opt' indicates whether the optimization algorithm found a feasible solution (○), or not (×).

#	Param. λ_{oc}	$\overline{x}$ α	β	FP	FN	SN α	β	FP	FN	SR α	β	FP	FN
1	1/12.5	0.095	0.802	0.099 (0.002)	0.838 (0.024)	0.062	0.938	0.062 (0.002)	0.888 (0.020)	0.098	0.902	0.174 (0.005)	0.661 (0.053)
2	1/15	0.037	0.805	0.062 (0.001)	0.841 (0.023)	0.062	0.938	0.062 (0.002)	0.838 (0.029)	0.098	0.902	0.174 (0.005)	0.513 (0.067)
3	1/17.5	0.022	0.764	0.049 (0.001)	0.818 (0.024)	0.062	0.938	0.062 (0.002)	0.789 (0.036)	0.098	0.902	0.174 (0.005)	0.394 (0.074)
4	1/20	0.015	0.719	0.042 (0.001)	0.79 (0.029)	0.062	0.938	0.062 (0.002)	0.742 (0.043)	0.098	0.902	0.174 (0.005)	0.301 (0.071)

Table B.20: Exponential distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

#	Param. λ_{oc}	$\overline{x}$ Exp. cost	$\overline{x}$ Cost	SN Exp. cost	SN Cost	SR Exp. cost	SR Cost
1	1/12.5	3,467.6	3,483.1 (47.9)	4,405.5	3,871.7 (169.2)	4,723.9	3,618.5 (103.9)
2	1/15	3,415.4	3,447.7 (38.0)	4,405.5	3,672.5 (116.5)	4,723.9	3,486 (68.7)
3	1/17.5	3,383.7	3,417.9 (34.4)	4,405.5	3,570.1 (91.3)	4,723.9	3,424.8 (53.6)
4	1/20	3,363.9	3,396.5 (33.0)	4,405.5	3,508.8 (75.5)	4,723.9	3,394.1 (45.6)

Table B.21: Exponential distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.

#	Param.		$\overline{x}$				SN				SR			
	μ_{oc}	b_{oc}	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*
1	12.5	$1/\sqrt{2}$	○	11.72	2	0.48	○	3	4	0.92	○	20	8	1.84
2	12.5	$2/\sqrt{2}$	○	12.4	2	0.46	○	3	4	0.92	○	20	8	1.84
3	12.5	$3/\sqrt{2}$	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
4	12.5	$4/\sqrt{2}$	○	13.62	2	0.46	○	3	4	0.92	○	20	8	1.84
5	12.5	$5/\sqrt{2}$	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
6	15	$1/\sqrt{2}$	○	13	1	0.5	○	3	4	0.92	○	20	8	1.84
7	15	$2/\sqrt{2}$	○	13.43	2	0.48	○	3	4	0.92	○	20	8	1.84
8	15	$3/\sqrt{2}$	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
9	15	$4/\sqrt{2}$	○	14.79	2	0.46	○	3	4	0.92	○	20	8	1.84
10	15	$5/\sqrt{2}$	○	15.37	2	0.46	○	3	4	0.92	○	20	8	1.84
11	17.5	$1/\sqrt{2}$	○	14.09	1	0.5	○	3	4	0.92	○	20	8	1.84
12	17.5	$2/\sqrt{2}$	○	15.01	1	0.49	○	3	4	0.92	○	20	8	1.84
13	17.5	$3/\sqrt{2}$	○	15.15	2	0.48	○	3	4	0.92	○	20	8	1.84
14	17.5	$4/\sqrt{2}$	○	15.91	2	0.46	○	3	4	0.92	○	20	8	1.84
15	17.5	$5/\sqrt{2}$	○	16.58	2	0.46	○	3	4	0.92	○	20	8	1.84
16	20	$1/\sqrt{2}$	○	15.25	1	0.5	○	3	4	0.92	○	20	8	1.84
17	20	$2/\sqrt{2}$	○	16	1	0.5	○	3	4	0.92	○	20	8	1.84
18	20	$3/\sqrt{2}$	○	17.06	1	0.47	○	3	4	0.92	○	20	8	1.84
19	20	$4/\sqrt{2}$	○	16.86	2	0.48	○	3	4	0.92	○	20	8	1.84
20	20	$5/\sqrt{2}$	○	17.65	2	0.46	○	3	4	0.92	○	20	8	1.84

Table B.22: Double exponential distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column 'Opt' indicates whether the optimization algorithm found a feasible solution (○), or not (×).

#	Param.		$\overline{x}$				SN				SR			
	μ_{oc}	b_{oc}	α	β	FP	FN	α	β	FP	FN	α	β	FP	FN
1	12.5	$1/\sqrt{2}$	0.008	0.134	0.013 (0.001)	0.116 (0.040)	0.062	0.938	0.063 (0.002)	0.055 (0.040)	0.098	0.902	0.098 (0.004)	0 (0.002)
2	12.5	$2/\sqrt{2}$	0.045	0.471	0.045 (0.001)	0.459 (0.057)	0.062	0.938	0.063 (0.002)	0.293 (0.073)	0.098	0.902	0.098 (0.004)	0.033 (0.032)
3	12.5	$3/\sqrt{2}$	0.081	0.586	0.073 (0.002)	0.602 (0.050)	0.062	0.938	0.063 (0.002)	0.481 (0.067)	0.098	0.902	0.098 (0.004)	0.149 (0.060)
4	12.5	$4/\sqrt{2}$	0.1	0.655	0.088 (0.002)	0.681 (0.044)	0.062	0.938	0.063 (0.002)	0.597 (0.060)	0.098	0.902	0.098 (0.004)	0.285 (0.074)
5	12.5	$5/\sqrt{2}$	0.1	0.717	0.088 (0.002)	0.748 (0.037)	0.062	0.938	0.063 (0.002)	0.672 (0.050)	0.098	0.902	0.098 (0.004)	0.399 (0.069)
6	15	$1/\sqrt{2}$	0.001	0.023	0.007 (0.001)	0.028 (0.022)	0.062	0.938	0.063 (0.002)	0.001 (0.006)	0.098	0.902	0.098 (0.004)	0 (0.000)
7	15	$2/\sqrt{2}$	0.008	0.134	0.013 (0.001)	0.116 (0.040)	0.062	0.938	0.063 (0.002)	0.055 (0.040)	0.098	0.902	0.098 (0.004)	0 (0.002)
8	15	$3/\sqrt{2}$	0.025	0.347	0.029 (0.001)	0.316 (0.054)	0.062	0.938	0.063 (0.002)	0.171 (0.063)	0.098	0.902	0.098 (0.004)	0.005 (0.013)
9	15	$4/\sqrt{2}$	0.045	0.471	0.045 (0.001)	0.459 (0.057)	0.062	0.938	0.063 (0.002)	0.293 (0.073)	0.098	0.902	0.098 (0.004)	0.033 (0.032)
10	15	$5/\sqrt{2}$	0.064	0.541	0.061 (0.002)	0.549 (0.049)	0.062	0.938	0.063 (0.002)	0.393 (0.069)	0.098	0.902	0.098 (0.004)	0.083 (0.049)
11	17.5	$1/\sqrt{2}$	0	0	0.002 (0.000)	0.004 (0.008)	0.062	0.938	0.063 (0.002)	0 (0.001)	0.098	0.902	0.098 (0.004)	0 (0.000)
12	17.5	$2/\sqrt{2}$	0.006	0.107	0.014 (0.001)	0.086 (0.035)	0.062	0.938	0.063 (0.002)	0.01 (0.018)	0.098	0.902	0.098 (0.004)	0 (0.000)
13	17.5	$3/\sqrt{2}$	0.008	0.134	0.013 (0.001)	0.116 (0.040)	0.062	0.938	0.063 (0.002)	0.055 (0.040)	0.098	0.902	0.098 (0.004)	0 (0.002)
14	17.5	$4/\sqrt{2}$	0.018	0.287	0.024 (0.001)	0.255 (0.053)	0.062	0.938	0.063 (0.002)	0.129 (0.059)	0.098	0.902	0.098 (0.004)	0.002 (0.008)
15	17.5	$5/\sqrt{2}$	0.031	0.397	0.035 (0.001)	0.368 (0.055)	0.062	0.938	0.063 (0.002)	0.214 (0.068)	0.098	0.902	0.098 (0.004)	0.011 (0.018)
16	20	$1/\sqrt{2}$	0	0	0 (0.000)	0.001 (0.003)	0.062	0.938	0.063 (0.002)	0 (0.000)	0.098	0.902	0.098 (0.004)	0 (0.000)
17	20	$2/\sqrt{2}$	0.001	0.023	0.007 (0.001)	0.028 (0.022)	0.062	0.938	0.063 (0.002)	0.001 (0.006)	0.098	0.902	0.098 (0.004)	0 (0.000)
18	20	$3/\sqrt{2}$	0.009	0.163	0.018 (0.001)	0.122 (0.040)	0.062	0.938	0.063 (0.002)	0.018 (0.024)	0.098	0.902	0.098 (0.004)	0 (0.000)
19	20	$4/\sqrt{2}$	0.008	0.134	0.013 (0.001)	0.116 (0.040)	0.062	0.938	0.063 (0.002)	0.055 (0.040)	0.098	0.902	0.098 (0.004)	0 (0.002)
20	20	$5/\sqrt{2}$	0.015	0.253	0.021 (0.001)	0.217 (0.050)	0.062	0.938	0.063 (0.002)	0.11 (0.055)	0.098	0.902	0.098 (0.004)	0.001 (0.007)

Table B.23: Double exponential distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

#	Param.		$\overline{x}$		SN		SR	
	μ_{oc}	b_{oc}	Exp. cost	Cost	Exp. cost	Cost	Exp. cost	Cost
1	12.5	$1/\sqrt{2}$	3,316.1	3,282.7 (18.4)	4,405.5	3,299.5 (23.4)	4,723.9	3,322.2 (29.0)
2	12.5	$2/\sqrt{2}$	3,363	3,324.7 (25.8)	4,405.5	3,325.2 (29.0)	4,723.9	3,326 (30.4)
3	12.5	$3/\sqrt{2}$	3,401.6	3,365 (30.9)	4,405.5	3,364.2 (37.4)	4,723.9	3,346.5 (34.9)
4	12.5	$4/\sqrt{2}$	3,428.9	3,394.1 (35.1)	4,405.5	3,406 (48.4)	4,723.9	3,379.6 (44.0)
5	12.5	$5/\sqrt{2}$	3,451.9	3,426.1 (42.5)	4,405.5	3,447.1 (60.7)	4,723.9	3,421 (53.6)
6	15	$1/\sqrt{2}$	3,289.7	3,274.5 (18.0)	4,405.5	3,293.5 (21.0)	4,723.9	3,321.2 (28.9)
7	15	$2/\sqrt{2}$	3,316.1	3,282.7 (18.4)	4,405.5	3,299.5 (23.4)	4,723.9	3,322.2 (29.0)
8	15	$3/\sqrt{2}$	3,339.1	3,301.7 (21.4)	4,405.5	3,309 (25.1)	4,723.9	3,321.8 (27.7)
9	15	$4/\sqrt{2}$	3,363	3,324.7 (25.8)	4,405.5	3,325.2 (29.0)	4,723.9	3,326 (30.4)
10	15	$5/\sqrt{2}$	3,384	3,346.4 (28.7)	4,405.5	3,343.2 (32.5)	4,723.9	3,335.1 (31.7)
11	17.5	$1/\sqrt{2}$	3,288.2	3,269.8 (18.5)	4,405.5	3,294.4 (21.6)	4,723.9	3,321.2 (28.9)
12	17.5	$2/\sqrt{2}$	3,295.9	3,281 (18.6)	4,405.5	3,296.5 (22.0)	4,723.9	3,321.2 (28.9)
13	17.5	$3/\sqrt{2}$	3,316.1	3,282.7 (18.4)	4,405.5	3,299.5 (23.4)	4,723.9	3,322.2 (29.0)
14	17.5	$4/\sqrt{2}$	3,330.9	3,296.1 (21.0)	4,405.5	3,305.1 (23.3)	4,723.9	3,322.2 (29.4)
15	17.5	$5/\sqrt{2}$	3,347.2	3,309.4 (22.1)	4,405.5	3,314.5 (26.5)	4,723.9	3,324 (28.6)
16	20	$1/\sqrt{2}$	3,288.1	3,270.2 (19.2)	4,405.5	3,294.4 (21.6)	4,723.9	3,321.2 (28.9)
17	20	$2/\sqrt{2}$	3,289.7	3,274.5 (18.0)	4,405.5	3,293.5 (21.0)	4,723.9	3,321.2 (28.9)
18	20	$3/\sqrt{2}$	3,300.4	3,284.4 (19.0)	4,405.5	3,295.7 (21.4)	4,723.9	3,321.2 (28.9)
19	20	$4/\sqrt{2}$	3,316.1	3,282.7 (18.4)	4,405.5	3,299.5 (23.4)	4,723.9	3,322.2 (29.0)
20	20	$5/\sqrt{2}$	3,326.9	3,292.8 (19.3)	4,405.5	3,303.2 (22.6)	4,723.9	3,319.4 (29.4)

Table B.24: Double exponential distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.

#	Param.		$\overline{x}$				SN				SR			
	p	a	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*
1	0.25	2	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
2	0.25	4	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
3	0.25	6	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
4	0.5	2	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
5	0.5	4	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
6	0.5	6	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
7	0.75	2	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
8	0.75	4	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84
9	0.75	6	○	12.96	2	0.46	○	3	4	0.92	○	20	8	1.84

Table B.25: Mixture of normals ($\mu_{\text{oc}} = 12.5, \sigma = 3$) distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column ‘Opt’ indicates whether the optimization algorithm found a feasible solution ($\circ$), or not ($\times$).

#	Param.		$\overline{x}$				SN				SR			
	p	a	α	β	FP	FN	α	β	FP	FN	α	β	FP	FN
1	0.25	2	0.081	0.586	0.089 (0.002)	0.595 (0.050)	0.062	0.938	0.062 (0.002)	0.539 (0.063)	0.098	0.902	0.117 (0.004)	0.121 (0.056)
2	0.25	4	0.081	0.586	0.09 (0.002)	0.599 (0.049)	0.062	0.938	0.063 (0.002)	0.283 (0.072)	0.098	0.902	0.162 (0.005)	0.009 (0.018)
3	0.25	6	0.081	0.586	0.077 (0.002)	0.58 (0.050)	0.062	0.938	0.063 (0.002)	0.085 (0.049)	0.098	0.902	0.167 (0.005)	0 (0.001)
4	0.5	2	0.081	0.586	0.084 (0.002)	0.581 (0.047)	0.062	0.938	0.063 (0.002)	0.627 (0.055)	0.098	0.902	0.098 (0.004)	0.203 (0.066)
5	0.5	4	0.081	0.586	0.097 (0.002)	0.592 (0.048)	0.062	0.938	0.062 (0.002)	0.72 (0.044)	0.098	0.902	0.098 (0.004)	0.277 (0.070)
6	0.5	6	0.081	0.586	0.108 (0.002)	0.623 (0.047)	0.062	0.938	0.063 (0.002)	0.781 (0.038)	0.098	0.902	0.098 (0.004)	0.363 (0.070)
7	0.75	2	0.081	0.586	0.074 (0.002)	0.563 (0.052)	0.062	0.938	0.062 (0.002)	0.632 (0.057)	0.098	0.902	0.082 (0.004)	0.267 (0.072)
8	0.75	4	0.081	0.586	0.047 (0.001)	0.515 (0.053)	0.062	0.938	0.063 (0.002)	0.69 (0.047)	0.098	0.902	0.06 (0.003)	0.446 (0.071)
9	0.75	6	0.081	0.586	0.022 (0.001)	0.469 (0.054)	0.062	0.938	0.062 (0.002)	0.679 (0.050)	0.098	0.902	0.059 (0.003)	0.552 (0.062)

Table B.26: Mixture of normals ($\mu_{\text{oc}} = 12.5, \sigma = 3$) distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

#	Param. p	Param. a	$\overline{x}$ Exp. cost	$\overline{x}$ Cost	SN Exp. cost	SN Cost	SR Exp. cost	SR Cost
1	0.25	2	3,401.6	3,371.3 (31.1)	4,405.5	3,379.8 (42.2)	4,723.9	3,343.5 (32.7)
2	0.25	4	3,401.6	3,374.4 (29.3)	4,405.5	3,323.4 (28.6)	4,723.9	3,329.3 (29.9)
3	0.25	6	3,401.6	3,362.9 (30.7)	4,405.5	3,300.5 (23.0)	4,723.9	3,329.9 (28.9)
4	0.5	2	3,401.6	3,365.6 (28.9)	4,405.5	3,418 (51.3)	4,723.9	3,358.8 (37.8)
5	0.5	4	3,401.6	3,375.6 (29.7)	4,405.5	3,487.2 (68.3)	4,723.9	3,377.1 (43.3)
6	0.5	6	3,401.6	3,388.3 (32.4)	4,405.5	3,558.5 (90.8)	4,723.9	3,405 (49.8)
7	0.75	2	3,401.6	3,357.9 (28.5)	4,405.5	3,423.2 (55.4)	4,723.9	3,375.3 (43.1)
8	0.75	4	3,401.6	3,335.7 (28.1)	4,405.5	3,461.8 (63.2)	4,723.9	3,437.7 (59.0)
9	0.75	6	3,401.6	3,315.2 (24.4)	4,405.5	3,453.1 (61.1)	4,723.9	3,497.2 (78.0)

Table B.27: Mixture of normals ($\mu_{\text{oc}} = 12.5, \sigma = 3$) distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.

#	Param.		$\overline{x}$				SN				SR			
	p	a	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*
1	0.25	2	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
2	0.25	4	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
3	0.25	6	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
4	0.5	2	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
5	0.5	4	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
6	0.5	6	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
7	0.75	2	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
8	0.75	4	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84
9	0.75	6	○	14.53	2	0.46	○	3	4	0.92	○	20	8	1.84

Table B.28: Mixture of normals ($\mu_{\text{oc}} = 12.5, \sigma = 5$) distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column 'Opt' indicates whether the optimization algorithm found a feasible solution (○), or not (×).

#	Param.		$\overline{x}$				SN				SR			
	p	a	α	β	FP	FN	α	β	FP	FN	α	β	FP	FN
1	0.25	2	0.1	0.717	0.107 (0.002)	0.72 (0.039)	0.062	0.938	0.062 (0.002)	0.747 (0.042)	0.098	0.902	0.117 (0.004)	0.421 (0.070)
2	0.25	4	0.1	0.717	0.107 (0.002)	0.702 (0.041)	0.062	0.938	0.063 (0.002)	0.611 (0.059)	0.098	0.902	0.163 (0.005)	0.193 (0.064)
3	0.25	6	0.1	0.717	0.09 (0.002)	0.67 (0.044)	0.062	0.938	0.063 (0.002)	0.41 (0.071)	0.098	0.902	0.167 (0.005)	0.042 (0.036)
4	0.5	2	0.1	0.717	0.104 (0.002)	0.706 (0.043)	0.062	0.938	0.063 (0.002)	0.792 (0.036)	0.098	0.902	0.098 (0.004)	0.513 (0.067)
5	0.5	4	0.1	0.717	0.122 (0.002)	0.71 (0.041)	0.062	0.938	0.063 (0.002)	0.871 (0.022)	0.098	0.902	0.098 (0.004)	0.561 (0.062)
6	0.5	6	0.1	0.717	0.144 (0.002)	0.732 (0.037)	0.062	0.938	0.062 (0.002)	0.913 (0.017)	0.098	0.902	0.098 (0.004)	0.603 (0.059)
7	0.75	2	0.1	0.717	0.094 (0.002)	0.7 (0.041)	0.062	0.938	0.062 (0.002)	0.779 (0.039)	0.098	0.902	0.082 (0.004)	0.562 (0.063)
8	0.75	4	0.1	0.717	0.071 (0.002)	0.653 (0.045)	0.062	0.938	0.063 (0.002)	0.761 (0.039)	0.098	0.902	0.06 (0.003)	0.677 (0.050)
9	0.75	6	0.1	0.717	0.042 (0.001)	0.603 (0.050)	0.062	0.938	0.063 (0.002)	0.718 (0.047)	0.098	0.902	0.059 (0.003)	0.675 (0.049)

Table B.29: Mixture of normals ($\mu_{\text{oc}} = 12.5, \sigma = 5$) distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

#	Param.		$\overline{x}$		SN		SR	
	p	a	Exp. cost	Cost	Exp. cost	Cost	Exp. cost	Cost
1	0.25	2	3,451.9	3,421.5 (39.9)	4,405.5	3,515.6 (79.4)	4,723.9	3,429.6 (56.2)
2	0.25	4	3,451.9	3,416.1 (38.9)	4,405.5	3,413 (50.5)	4,723.9	3,363.7 (37.3)
3	0.25	6	3,451.9	3,394.7 (35.8)	4,405.5	3,346.4 (33.9)	4,723.9	3,336.9 (28.5)
4	0.5	2	3,451.9	3,414.9 (40.2)	4,405.5	3,576.2 (90.6)	4,723.9	3,479.1 (70.0)
5	0.5	4	3,451.9	3,427.8 (39.9)	4,405.5	3,789.2 (143.2)	4,723.9	3,509.1 (76.1)
6	0.5	6	3,451.9	3,447.7 (43.8)	4,405.5	4,055.4 (217.8)	4,723.9	3,545 (86.5)
7	0.75	2	3,451.9	3,407.8 (38.2)	4,405.5	3,557.6 (87.1)	4,723.9	3,507.9 (77.4)
8	0.75	4	3,451.9	3,380.6 (36.1)	4,405.5	3,530.1 (80.8)	4,723.9	3,628.8 (107.7)
9	0.75	6	3,451.9	3,346.8 (32.2)	4,405.5	3,486.8 (71.6)	4,723.9	3,623.4 (105.3)

Table B.30: Mixture of normals ($\mu_{\text{oc}} = 12.5, \sigma = 5$) distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.

	Param.		$\overline{x}$				SN				SR			
#	p	a	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*	Opt	k^*	n^*	h^*
1	0.25	2	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
2	0.25	4	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
3	0.25	6	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
4	0.5	2	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
5	0.5	4	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
6	0.5	6	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
7	0.75	2	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
8	0.75	4	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84
9	0.75	6	○	14.17	2	0.46	○	3	4	0.92	○	20	8	1.84

Table B.31: Mixture of normals ($\mu_{\mathrm{oc}} = 15, \sigma = 3$) distribution. Results of the optimization of the economic statistical design for every considered control chart. This table gives the optimal values of k, n, and h, for every experimental configuration. The column 'Opt' indicates whether the optimization algorithm found a feasible solution (○), or not (×).

	Param.		$\overline{x}$				SN				SR			
#	p	a	α	β	FP	FN	α	β	FP	FN	α	β	FP	FN
1	0.25	2	0.025	0.347	0.031 (0.001)	0.361 (0.054)	0.062	0.938	0.063 (0.002)	0.097 (0.051)	0.098	0.902	0.117 (0.004)	0 (0.001)
2	0.25	4	0.025	0.347	0.041 (0.001)	0.413 (0.053)	0.062	0.938	0.063 (0.002)	0.005 (0.013)	0.098	0.902	0.162 (0.005)	0 (0.000)
3	0.25	6	0.025	0.347	0.051 (0.001)	0.474 (0.054)	0.062	0.938	0.063 (0.002)	0 (0.001)	0.098	0.902	0.167 (0.005)	0 (0.000)
4	0.5	2	0.025	0.347	0.023 (0.001)	0.347 (0.057)	0.062	0.938	0.063 (0.002)	0.158 (0.062)	0.098	0.902	0.098 (0.004)	0.001 (0.006)
5	0.5	4	0.025	0.347	0.015 (0.001)	0.335 (0.054)	0.062	0.938	0.063 (0.002)	0.077 (0.048)	0.098	0.902	0.098 (0.004)	0 (0.001)
6	0.5	6	0.025	0.347	0.006 (0.001)	0.301 (0.055)	0.062	0.938	0.063 (0.002)	0.021 (0.026)	0.098	0.902	0.098 (0.004)	0 (0.000)
7	0.75	2	0.025	0.347	0.017 (0.001)	0.334 (0.056)	0.062	0.938	0.062 (0.002)	0.254 (0.071)	0.098	0.902	0.082 (0.004)	0.006 (0.014)
8	0.75	4	0.025	0.347	0.003 (0.000)	0.343 (0.054)	0.062	0.938	0.063 (0.002)	0.472 (0.069)	0.098	0.902	0.06 (0.003)	0.037 (0.034)
9	0.75	6	0.025	0.347	0 (0.000)	0.378 (0.056)	0.062	0.938	0.063 (0.002)	0.593 (0.058)	0.098	0.902	0.059 (0.003)	0.083 (0.048)

Table B.32: Mixture of normals ($\mu_{\text{oc}} = 15, \sigma = 3$) distribution. Results of the simulation experiments where the expected probabilities of type I (α) and type II (β) errors are compared to the probabilities observed during the experiments. 'FP' represents the false positives rates (type I errors), and 'FN' represents the false negatives rates (type II errors) observed during the simulations. The values between brackets are the corresponding standard deviations.

#	Param.		$\overline{x}$		SN		SR	
	p	a	Exp. cost	Cost	Exp. cost	Cost	Exp. cost	Cost
1	0.25	2	3,339.1	3,307.1 (22.0)	4,405.5	3,301.4 (23.3)	4,723.9	3,323.1 (28.8)
2	0.25	4	3,339.1	3,318.7 (23.2)	4,405.5	3,295.4 (22.9)	4,723.9	3,329.2 (28.9)
3	0.25	6	3,339.1	3,330.8 (26.4)	4,405.5	3,294.4 (22.5)	4,723.9	3,329.9 (28.9)
4	0.5	2	3,339.1	3,302.7 (22.3)	4,405.5	3,308.4 (24.9)	4,723.9	3,321.4 (29.2)
5	0.5	4	3,339.1	3,297.7 (21.6)	4,405.5	3,300 (23.5)	4,723.9	3,320.5 (28.9)
6	0.5	6	3,339.1	3,288.5 (21.4)	4,405.5	3,297.4 (22.5)	4,723.9	3,320.5 (28.9)
7	0.75	2	3,339.1	3,298.4 (22.6)	4,405.5	3,320.6 (26.8)	4,723.9	3,318.4 (28.8)
8	0.75	4	3,339.1	3,290.6 (22.4)	4,405.5	3,361.5 (36.1)	4,723.9	3,322.3 (30.1)
9	0.75	6	3,339.1	3,292.2 (23.1)	4,405.5	3,402.2 (47.5)	4,723.9	3,329.2 (32.8)

Table B.33: Mixture of normals ($\mu_{\text{oc}} = 15, \sigma = 3$) distribution. Results of the simulation experiments for every considered control chart. This table compares the expected cost of operating the process given by the cost model and the actual average cost obtained during the experiments. The values between brackets are the corresponding standard deviations.