S-adic conjecture and diagrams

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Abstract

In this note we apply a substantial improvement of a result of S. Ferenczi on S-adic subshifts to give Bratteli-Vershik representations of these subshifts.

Résumé

Dans cette note nous utilisons une amélioration conséquente d'un résultat de S. Ferenczi, concernant les sous-shifts S-adiques, afin d'en trouver des représentations de Bratteli-Vershik.

Version française abrégée

Il est désormais bien connu que si un sous-shift (X, T) minimal est de complexité sous-affine (i.e., $(p_X(n)/n)_n$ est bornée, où $p_X(n)$ est le nombre de mots de longueur n apparaissant dans les suites de X), alors X est obtenu par un produit infini de morphismes appartenant à un ensemble fini S. C'est un résultat prouvé par S. Ferenczi dans [6]. On dit alors que (X, T) est S-adique. Préalablement à ce résultat B. Host a conjecturé qu'il existait une notion de S-adicité forte équivalente à la sous-affinité de la complexité. La preuve de Ferenczi (plus précisément, sa présentation) ne permet pas d'en extraire sans peine une telle notion. Dans un travail récent ([9]) le second auteur a retravaillé cette preuve afin d'y parvenir. Ceci a été réalisé dans [10] pour un cas particulier : il existe une notion de forte S-adicité équivalente à être minimal et tel que $(p_X(n+1)-p_X(n))_n$ est bornée par 2. Cette notion est relative à un ensemble S constitué de 5 morphismes propre à cette classe de complexité. Lorsque $(p_X(n+1)-p_X(n))_n$ est bornée par 2 à partir de k (propriété notée $\Pi(2,k)$), alors un morphisme supplémentaire est nécessaire. Ferenczi avait obtenu ce type de résultat avec un ensemble S de cardinal borné par 3^{27}.

Dans cette note nous présentons une application de ce résultat à la représentation des sous-shifts S-adiques par des diagrammes de Bratteli. Rappelons que ces diagrammes, introduits dans le contexte des systèmes dynamiques par A. Vershik [15], sont au coeur de la caractérisation algébrique (par les groupes de

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dimension) de l'équivalence orbitale des actions minimales de $\mathbb{Z}$ sur des Cantor [8,7]. Ces représentations sont très utiles dans les problèmes liés aux phénomèmes de récurrence. Mais, étant donnée une action de $\mathbb{Z}$, il est en général difficile d'en trouver une représentation de Bratteli-Vershik "canonique" (voir [4] pour des exemples). C'est ce que nous présentons ici pour les sous-shifts minimaux vérifiant $\Pi(2,k)$. Pour cela nous mettons en place quelques résultats de combinatoires sur les mots, les morphismes et les diagrammes de Bratteli qui ont un spectre d'applications plus large que celui présenté ici. Ils permettent, étant donnés des morphismes bien choisis représentant le sous-shift, d'en donner, sans plus de travail, une représentation de Bratteli-Vershik.

Pour bien saisir le cas S-adique, nous illustrons ces remarques combinatoires au cas des sous-shifts substitutifs [5]. En conséquence, nous obtenons un résultat de représentation des sous-shifts, Corollary 2.3. A l'aide d'un théorème de T. Downarowicz et A. Maass [3], énonçant que les actions minimales de $\mathbb{Z}$ sur des Cantor de rang topologique fini sont des sous-shifts ou des odomètres, nous déduisons le Corollary 2.5 que nous appliquons aux sous-shifts S-adiques. Dans la Section 3 nous énonçons le résultat de Leroy [10] par lequel nous obtenons des représentations de Bratteli-Vershik pour les sous-shifts minimaux (X,T) satisfaisant à $\Pi(2,k)$.

1. Introduction

In their seminal papers [12,13] G. A. Hedlund and M. Morse proved that a sequence $x \in A^{\mathbb{N}}$ is ultimately periodic if and only if $p_x(n) = n$ for some n where $p_x(n)$ is the word complexity of x, that is the number of distinct words of length n in x. Moreover they showed that sequences satisfying $p_x(n) = n+1$ for all n exist, are uniformly recurrent, intimately related to the rotations on the torus and S-adic, i.e., produced by an infinite product of finitely many morphisms (3 in fact). Then, certainly induced by the (sub-affine complexity) examples in [1] and the complexity of substitutions ([14]), B. Host conjectured that there exists a strong notion of S-adicity which is equivalent to sub-affine complexity. With the help of the nice result of J. Cassaigne [2] showing that a sequence x has sub-affine complexity if and only if $(p_x(n+1) - p_x(n))_n$ is bounded, Ferenczi proved in [6] that minimal subshift with sub-affine complexity (i.e., $(p_X(n)/n)_n$ is bounded, where $p_X(n)$ is the number of words of length n appearing in sequences of X) are S-adic (i.e., obtained by an infinite product of finitely many morphisms). And, in the case it is ultimately bounded by 2, he showed that less than 3^{27} morphisms are needed. The technicality of the proof did not allow to imagine and define the "strong notion of S-adicity" that is looked for. In [9] the second author presented a more detailed proof of Ferenczi's result that leaded to some improvements (see also [11]). Following Ferenczi's approach he showed that there exists a set S consisting of 5 morphisms (to be compared to 3^{27}) such that when $(p_X(n+1) - p_X(n))_n$ is bounded by 2 then X is described by an infinite product of morphisms belonging to S. This provides a "strong notion of S-adicity" which is equivalent to this property. When (X,T) satisfies the property $\Pi(2,k)$ to have $(p_X(n+1) - p_X(n))_n$ is bounded by 2 from $n = k$, then an other morphism is needed. We present this result in this note (Theorem 3.1) with a statement that applies to Bratteli-Vershik representation of such subshifts. We recall that Bratteli-Vershik representations of minimal $\mathbb{Z}$-actions on Cantor sets are powerful tools that were used to solve the topological orbit equivalence relation [7] and that are very useful to solve problems where recurrence properties are involved. Observe that it is usually not so easy to find such a representation and difficult to find the "canonical" representation. Nevertheless it has been done for various classical family of dynamical systems such that substitutive subshifts, Toeplitz subshifts, interval exchange transformations, sturmian subshifts, odometers and linearly recurrent subshifts. In this note we present such representations for subshifts satisfying $\Pi(2,k)$.

In Section 2 we recall the definition of Bratteli diagram and Bratteli-Vershik representation. Then we present some combinatorics on words and morphisms that notably helps to obtain such representations of subshifts (Corollary 2.5). In Section 3, we present one of the main result of Leroy in [10] and, as an application of this result, we deduce BV-representations of S-adic subshifts (X, T) satisfying $\Pi(2, k)$.

2. Bratteli-Vershik representation of S-adic subshifts

2.1. *Bratteli-Vershik representations and subshifts*

Consider a minimal Cantor dynamical system (X, T) (MCDS, also called minimal $\mathbb{Z}$-action on a Cantor set), i.e., a homeomorphism T on a compact metric zero-dimensional space X with no isolated points, such that the orbit $\{T^n x; n \in \mathbb{Z}\}$ of every point $x \in X$ is dense in X. We assume familiarity of the reader with the Bratteli-Vershik diagram representation of such systems, yet we recall it briefly in order to establish the notation. For more details see [8,4]. The vertices of the Bratteli-Vershik diagram $\mathcal{B} = (V, E)$ are organized into countably many finite subsets of vertices V_i, $i \geq 0$ (where V_0 is a singleton $\{v_0\}$) and subsets of edges E_i, $i \geq 1$. Thus, $V = \cup_{i \geq 0} V_i$ and $E = \cup_{i \geq 1} E_i$. Every edge $e \in E_{i+1}$ connects a vertex $s = s(e) \in V_{i+1}$ for some $i \geq 0$ with some vertex $t = t(e) \in V_i$. At least one edge goes upward and at least one goes downward from each vertex in V_{i+1}. Multiple arrows connecting the same vertices are admitted. We assume that the diagram is *simple*, i.e., that there is a subsequence $(i_k)_{k \geq 0}$ such that from every vertex in $V_{i_{k+1}}$ there is an upward path (going upward at each level) to every vertex in V_{i_k}. For each vertex $v \in V_i$ (except v_0) the set of all edges $\{e_1, \ldots, e_l\}$ going upward (to V_{i-1}) from v is ordered linearly: let say $e_1 < \ldots < e_l$. It will be convenient to consider the morphisms $\sigma_i^{\mathcal{B}} : V_i^* \to V_{i-1}^*$ defined by $\sigma_i^{\mathcal{B}}(v) = t(e_1) \cdots t(e_l)$. We will say it is the morphism we read at level i on $\mathcal{B}$. In the sequel we will always suppose that $\mathcal{B}$ is such that $\sigma_1^{\mathcal{B}}(v) = v_0$ for all $v \in V_1$. This ordering induces a lexicographical order on all upward paths from v to v_0, and a partial order on all infinite upward paths arriving to v_0. We denote by $X_{\mathcal{B}}$ the set of all such infinite paths. We assume that this partial order has a unique minimal element x_m (i.e., such that all its edges are minimal for the local order) and a unique maximal one x_M (whose all edges are maximal for the local order). This defines a map $V_{\mathcal{B}}$ sending every element x to its successor in the partial order and sending x_M to x_m.

Theorem 2.1 ([8]) *Let (X, T) be a minimal Cantor dynamical system. Then, there exists a simple Bratteli diagram $\mathcal{B}$ such that (X, T) is topologically conjugate to $(X_{\mathcal{B}}, V_{\mathcal{B}})$.*

This representation theorem has been used in [7] to characterize the orbit equivalence relation of MCDS. We say that $(X_{\mathcal{B}}, V_{\mathcal{B}})$ is a BV-representation of (X, T).

In the sequel (X, T) will exclusively be a *subshift*: X is a T-invariant closed subset of $A^{\mathbb{Z}}$ (endowed with the product topology) where A is a finite alphabet and T is the shift map $(T((x_n)_n) = (x_{n+1})_n)$. Note that the shift map will always be denoted by T: we will specify neither X nor A. We say that (X, T) is generated by $x \in A^{\mathbb{Z}}$ when X is the set of sequences y such that for any i and j the word $y_i y_{i+1} y_{i+j}$ occurs in x. We set $[u.v]_X = \{x \in X | x_{-|u|} \cdots x_{|v|-1} = uv\}$ and we call such sets *cylinder sets*. They are clopen sets and form a base of the topology of X. When u is the empty word we write $[v]_X$.

2.2. *Proper morphisms, substitutions and BV-representations*

In the sequel A, B, A_n, … are finite alphabets and A^* denote the free monoid generated by A.

A morphism $\sigma : A^* \to B^*$ is *left proper* (resp. *right proper*) if there exists a letter $l \in B$ (resp. $r \in B$) such that for all $a \in A$, $\sigma(a) = l u(a)$ (resp. $\sigma(a) = u(a) r$) for some $u(a) \in B^*$. It is *proper* when it is both left and right proper. Let σ be left proper. The morphism $\tau : A^* \to B^*$ defined by $\tau(a) = u(a) l$ is the *left conjugate* of σ and it is right proper. In the same way we define the *right conjugate* of σ (it is left proper).

Lemma 2.2 *Let $\sigma : A^* \to B^*$ be a left proper (resp. right proper) morphism with first letter l (resp. last letter r) and τ be its left (resp. right) conjugate then for all $a \in A$ and n*

$$\sigma^n(a)l = l\tau^n(a) \quad (\textit{resp. } r\sigma^n(a) = \tau^n(a)r).$$

A substitution is an endomorphism $\sigma : A^* \to A^*$ such that there exists a letter a with $\sigma(a) = au$ where u is not the empty word. The subshift (X_σ, T) it generates consists of $x \in A^{\mathbb{Z}}$ such that all words $x_i x_{i+1} \cdots x_j$ of x have an occurrence in some $\sigma^n(a)$. We refer to [14] for more details.

To obtain a BV-representation of (X_σ, T), an idea (first developed in [15]) is to consider the Bratteli diagram $\mathcal{B}$ where the morphism we read at each level on $\mathcal{B}$ is σ. When (X_σ, T) is minimal this provides a measure-theoretical representation which is not necessarily topological. The problem with this construction is that minimal paths correspond to right fixed points and maximal paths to left fixed points of σ. Thus, to obtain such a topological representation, we need to have a unique fixed point in $A^{\mathbb{Z}}$.

In [5] is shown that to have a stationary BV-representation of (X_σ, T) (i.e., where the substitutions read on the E_i are equal up to some bijective changes of the alphabets), it suffices to find a proper substitution ζ such that (X_ζ, T) is isomorphic to (X_σ, T). Then the stationary BV-representation is given by the Bratteli diagram where at each level $i \geq 2$ we read ζ. Moreover an algorithm is given to find such ζ. Consequently to Lemma 2.2 the following proposition claims it is enough to have ζ left or right proper.

Proposition 2.1 *Let σ be a left or right proper primitive substitution and τ be a conjugate. Then,*

(i) $\sigma\tau$ and $\tau\sigma$ are proper primitive substitutions;

(ii) (X_σ, T), (X_τ, T), $(X_{\tau\sigma}, T)$, $(X_{\sigma\tau}, T)$, $(X_{\mathcal{B}_1}, V_{\mathcal{B}_1})$ and $(X_{\mathcal{B}_2}, V_{\mathcal{B}_2})$ are pairwise conjugate where $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is the stationary ordered Bratteli diagram we read $\sigma\tau$ (resp. $\tau\sigma$) on from level 2.

The algorithm used in [5] to find BV-representations of substitution subshifts has a simplified version that can be used to find a left (or right) proper substitution. This leads to BV-representations with less edges and vertices. In the sequel we develop the idea of Lemma 2.2 to a more general framework.

2.3. *Combinatorics on words and BV-representations*

Let S be a (possibly infinite) set of morphisms. An *S-adic representation* of a subshift (X, T) is a sequence $(\sigma_n, a_n)_{n \geq 2}$ where, for all n, $\sigma_n : A_n^* \to A_{n-1}^*$ belongs to S, a_n belongs to A_n and X is the set of sequences $x \in A_1^{\mathbb{Z}}$ such that all words $x_i x_{i+1} \cdots x_j$ appear in some $\sigma_2\sigma_3 \cdots \sigma_n(a_n)$. We start the representation with $n = 2$ in order to fit with the Bratteli diagram notation: σ_n will correspond to E_n from $n = 2$. When such a sequence is fixed, we denote by (X_n, T) the subshift generated by $(\sigma_k, a_k)_{k \geq n}$. Observe that X_n is included in $A_{n-1}^{\mathbb{Z}}$.

Proposition 2.2 *Let (X, T) be the minimal S-adic subshift defined by $(\sigma_n : A_n^* \to A_{n-1}^*, a_n)_n$ where the σ_n are proper. Suppose that for all n the morphisms σ_n extend by concatenation to a one-to-one map from X_n to X_{n-1}. Then, (X, T) is isomorphic to $(X_{\mathcal{B}}, V_{\mathcal{B}})$ where $\mathcal{B}$ is the Bratteli diagram such that for all $n \geq 2$ the substitution read on $\mathcal{B}$ at level n is σ_n.*

Proof. We can suppose that all the images of σ_n starts with the letter a and ends with b. Define, for all n, $\tau_n = \sigma_2 \cdots \sigma_n$ and (X_n, T) the subshift generated by $(\sigma_l : A_l^* \to A_{l-1}^*, a_l)_{l \geq n}$. Notice that $\tau_n(X_{n+1})$ is included in X and consider

$$\mathcal{P}(n) = \left\{ T^j \tau_n([c]_{X_{n+1}}) \mid c \in A_n, 0 \leq j < |\tau_n(c)| \right\} .$$

From [8], to conclude, it suffices to prove $(\mathcal{P}(n))_n$ is a nested sequence of partitions generating the topology of X satisfying $\# \cap_n \cup_{c \in A_n} \tau_n([c]_{X_{n+1}}) = 1$. Let prove it is a partition. Let $x \in X$. The maps $\sigma_n : X_n \to X_{n-1}$ being one-to-one, there exists a unique couple (y, j), $y = (y_l)_{l \in \mathbb{Z}} \in X_n$ and $j \in \{0, 1, \ldots, |\tau_n(y_0)| - 1\}$, such that $x = T^j \tau_n(y)$. Then, x belongs to $T^j \tau_n([y_0]_{X_n})$ and $\mathcal{P}(n)$ is a partition.

Now, let prove it is nested. Let $\Omega = T^j\tau_{n+1}([c]_{X_{n+2}})$ be an atom of $\mathcal{P}(n+1)$. Let $\sigma_{n+1}(c) = c_1 \cdots c_l$ and i such that $|\tau_n(c_1 \cdots c_i)| \leq j < |\tau_n(c_1 \cdots c_{i+1})|$. Then, $\Omega \subset T^{j-|\tau_n(c_1\cdots c_i)|}\tau_n([c_{i+1}]_{X_{n+1}})$ and $(\mathcal{P}(n))_n$ is nested. We let as an exercise to prove it generates the topology of X.

As $\tau_{n+1}([c]_{X_{n+2}}) \subset \tau_n([b.a]_{X_{n+1}})$, from the assumptions, we deduce $\cap_n \cup_{c\in A_n} \tau_n([c]_{X_{n+1}}) = 1$. $\square$

Observe that in Proposition 2.2, it is important to suppose that σ_n is one-to-one. For examples, if the σ_n were equal to σ, where $\sigma(a) = ab$ and $\sigma(b) = ab$, then (X,T) would be a two-point periodic subshift and $(X_\mathcal{B}, V_\mathcal{B})$ would be the 2-odometer. Using Lemma 2.2 we obtain the following corollary.

Corollary 2.3 *Let (X,T) be the minimal S-adic subshift defined by $(\sigma_n, a_n)_{n\geq 2}$ where the σ_n are left or right proper. Suppose that for all n the morphisms σ_n extend by concatenation to one-to-one maps from X_n to X_{n-1}. Then (X,T) is isomorphic to $(X_\mathcal{B}, V_\mathcal{B})$ where $\mathcal{B}$ is the Bratteli diagram where for all $n \geq 2$:*

(i) the substitution read on E_{2n} is left proper and equal to σ_{2n} or its conjugate;

(ii) the substitution read on E_{2n+1} is right proper and equal to σ_{2n+1} or its conjugate.

More can be said about these subshifts but we need the following theorem proved in [3]. We say that a MCDS (X,T) has *topological rank k* if k is the smallest integer such that (X,T) has a BV-representation $(X_\mathcal{B}, V_\mathcal{B})$ with $(|V(n)|)_n$ bounded by k. When such a k does not exist we say that it has infinite topological rank. Examples of such systems are the odometers ($k = 1$), all primitive substitutive subshifts, all sturmian subshifts ($k = 2$), all linearly recurrent subshifts, some toeplitz subshifts (see [4] for a survey).

Theorem 2.4 ([3]) *Let (X,S) be a minimal Cantor dynamical system with topological rank k. Then (X,S) is expansive if, and only if, $k \geq 2$. Otherwise it is equicontinuous.*

In fact, we can be more precise. Let $(X_\mathcal{B}, V_\mathcal{B})$ be a Bratteli-Vershik representation of (X,S) whose set of vertices of $\mathcal{B}$ is bounded by k. Now let (X_n, T) be the subshift generated by $(\sigma_k^\mathcal{B}, a_k)_{k\geq n}$, where a_k belongs to V_k. Then, if there exists n such that (X_n,T) is not periodic then (X,S) is a subshift, otherwise it is equicontinuous. From this theorem and Corollary 2.3 we deduce the following corollary that enables us to give a BV-representation of a subshift once we have a "nice" S-adic representation.

Corollary 2.5 *Let (X,T) be a MCDS with and $(\sigma_n : A_n^* \to A_{n-1}^*, a_n)_n$ be such that $(|A_n|)_n$ is bounded by k. The following are equivalent.*

(i) (X,T) is the non-periodic subshift defined by the sequence $(\sigma_n, a_n)_n$ satisfying: For all n the map $\sigma_n : X_n \to X_{n-1}$ is well-defined, one-to-one and left or right proper.

(ii) (X,T) is isomorphic to $(X_\mathcal{B}, V_\mathcal{B})$ where $\mathcal{B}$ is the Bratteli diagram verifying:

  (a) for all n, the substitution read on E_{2n} is left proper and equal to σ_{2n} or its conjugate;

  (b) for all n, the substitution read on E_{2n+1} is right proper and equal to σ_{2n+1} or its conjugate;

  (c) (X_2,T), defined by $(\sigma_k, a_k)_{k\geq 2}$, is not periodic.

Moreover, in these situations the topological rank of (X,T) is bounded by k.

In the next section we apply this corollary to the subshift having a first difference of block complexity less or equal to 2 in order to obtain their Bratteli-Vershik representations.

3. Applications to sub-affine complexity

The following result was proven in [9]. We adapt the statement to our context. The original statement gives a complete answer to the S-adic conjecture (see [6]) in a restricted context.

Theorem 3.1 *Let (X,T) be an aperiodic minimal subshift satisfying $\Pi(2,k)$, then (X,T) is an S-adic subshift defined by $(\sigma_n, a_n)_{n\geq 2}$ where for all $n \geq 3$, σ_n belongs to $S = \{D, G, E_{ab}, E_{bc}, M\}$ with*

$$D: a \mapsto ab,\quad G: a \mapsto ba,\quad E_{ab}: a \mapsto b,\quad E_{bc}: a \mapsto a,\quad M: a \mapsto a$$
$$\ b \mapsto b \qquad\quad b \mapsto b \qquad\qquad b \mapsto a \qquad\qquad b \mapsto c \qquad\qquad b \mapsto b$$
$$\ c \mapsto c \qquad\quad c \mapsto c \qquad\qquad c \mapsto c \qquad\qquad c \mapsto b \qquad\qquad c \mapsto b.$$

Moreover, $\sigma_n : X_n \to X_{n-1}$ is one-to-one and there exists an increasing sequence $(n_i)_i$ such that for all i, $\sigma_{n_i}\sigma_{n_i+1}\cdots\sigma_{n_{i+1}}$ is proper and all letters in $\{a,b,c\}$ occur in all images $\sigma_{n_i}\sigma_{n_i+1}\cdots\sigma_{n_{i+1}}(y)$, $y \in \{a,b,c\}$. Furthermore, σ_2 can be algorithmically determined

Theorem 3.1 is weaker than the one in [9]. Indeed, the complete result states that there is an effective and non trivial labelled directed graph $\mathcal{G}$ such that any sequence of morphisms $(\sigma_n)_n$ (in Theorem 3.1) labels an infinite path in $\mathcal{G}$. Some (computable) additional conditions also allow to characterize all possible sequences $(\sigma_n)_n$. The Bratteli-Vershik version of this result is the following.

Corollary 3.2 *With the assumptions and notations of Theorem 3.1, the topological rank of (X,T) is at most 3. More precisely, (X,T) is isomorphic to $(X_{\mathcal{B}}, V_{\mathcal{B}})$ where $\mathcal{B}$ is the Bratteli diagram such that, for all $n \leq 2$, the substitution read on E_n is σ_n.*

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